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# An optimization of nonlinear control system based on quantum-mechanical "superposition-principle" $\dagger$ 

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#### Abstract

For nonlinear Hamliton-Jacobi equation, we have proposed a new method to deal with a framework of linear theories. "Superposable complex waves" and a new constant $H_{R}$ representing "strength" of the wave are introduced in this paper. Attendant to an optimal path we can assume a superposable complex wave in the state space subject to a "linear wave equation". Within a weak limit of the wave the optimal path is obtained. This method is realized as an algorithm according to two factors; the optimal control system as a constrained dynamical system, and its "linear operator" formalism using "Dirac bracket". A real-part projection of the complex linear wave equation is shown to lead to a "generalized Hamilton-Jacobi equation", where a term related to the wave is added to the Hamilton-Jacobi equation. A state feedback scheme is derived from this algorithm and a nonlinear system with 1 -input 1 -state variable is optimized by a simulation study using a typical method for problems of complex linear wave equations.


Key Words: Hamilton-Jacobi equation, Dirac brackets, linear operator, constrained mechanics, nonlinear optimal control theory, quantum-mechanical superposition-principle

## 1. Introduction

A governing equation of optimization of nonlinear feedback control is Hamilton-Jacobi equation. A solution of the equation leads to nonlinear optimal feedback in closed loop in the most systematic way. However, the HamiltonJacobi equation is a nonlinear partial differential equation in both space and time. Various schemes including approximations have so far been proposed ${ }^{\left.1)^{2}\right)}$ 3) 4) to solve the equation, but there has been no ultimate method. While we have a way of strict linearization of coordinates, but there are restrictions peculiar to coordinate transformations and this method is not relevant to any system.
We start with a concept of "a wave represented by complex numbers". We then propose a method of research in a framework of "linear wave equations" for systems with nonlinearities in state equations and control specifications given as performance time integrals. An optimal path is a curve in state space as shown in Fig.1, starting at a point $P_{1}$ at time $t_{1}$ ending at a point $P_{2}$ at time $t_{2}$, on each point of which a manipulation of optimal feedback control is operated at each time. This curve of the optimal path is a solution of the nonlinear partial differential equation of Hamilton-Jacobi. We then form an idea of a wave $\psi$ laid out in a background of the optimal path as shown in Fig.2. And we introduce a new design constant $H_{R}$

[^0]to control wave strength acted on the path. The curve is deformed by the wave action, however it approaches the optimal path shown in Fig. 1 when the wave action is sufficiently weak. Because waves are superposable, a governing equation of the wave acting on the optimal path is linear. This suggests that we can approximate nonlinear feedback control by making use of a linear wave equation.

The paper is organized as follows. In Section 2, we formulate an optimal control system according to the conventional method ${ }^{7)}$ that state equations are regarded as constraint conditions. A wave in state space is also subject to these constraint conditions. We then pick up all the constraint conditions and clarify an algebraic structure among these constraint conditions. To describe optimal control systems we define canonical coordinates (state variables, control variables and costate variables) and corresponding canonical momenta. We define a Hamiltonian that describes development of the system in time. We then calculate conditions that the constraint conditions close under time development. After that to treat consistently the optimal control system as a constrained dynamical system, we introduce a "Dirac bracket" into a set of these canonical variables(canonical coordinates and canonical momenta). The Dirac bracket in constrained dynamical systems is an extension of the Poisson bracket. In terms of the Dirac bracket, the equation of motion of the constrained dynamical system is described without paying attention to the constraint conditions. We show the conventional form of the Hamilton-Jacobi equation in the last part of this section. In Section 3, we introduce "linear operators" which are in one-to-one correspondence to the canonical variables. Especially to the control variables correspond "control variable operators", and to the costate variables correspond "costate variable operators". These linear operators are defined ${ }^{8)}$ such that they satisfy all commutation relations when the Dirac brackets are replaced with commutation relations. From properties of these commutation relations, we can represent these linear operators as those operating on a linear space of state functions. For an affine system, we explicitly write down partial differential operators which represent these linear operators. Moreover, we can construct a "Hamiltonian operator" corresponding to the Hamiltonian of the optimal control system. Using this Hamiltonian operator we set up a linear wave equation imposed on a complexvalued "wave function". We can show for an affine system, that a phase function of the wave function satisfies an equation which is nothing but the Hamilton-Jacobi equation under an additional cost function representing
wave action. This "additional cost" due to wave action is a quantity proportional to the square of the control constant $H_{R}$. We call this equation for the phase function of the wave function the "generalized Hamilton-Jacobi equation". We then see that this equation approximates the Hamilton-Jacobi equation when we set the additional cost as small as possible by taking the control constant $H_{R}$ close to zero. State feedback laws are also given by the wave function and its partial differentials. In Section 4, to clarify the concepts we optimize a simple nonlinear system with 1 -input and 1 -state by using the corresponding wave function. A typical algorithm of perturbation theory is adopted to solve numerically the wave equation. We examine wave action to the optimal path taking three values of the control constant $H_{R}$. In this example, we show that the generalized Hamilton-Jacobi equation approximates the Hamilton-Jacobi equation. Summary and discussion are given in Section 5 .

## 2. Optimal control systems as constrained dynamical systems

According to the conventional method ${ }^{7)}$, we treat the state equation as the constraint condition. We then describe development of the optimal control system in time using a method pioneered by Dirac. ${ }^{8)}$ The reason of applying Dirac's method is that the control and costate variables are to be represented by the linear operators, even when there are the constraint condition. In this section, after we introduce the optimal control system, we use the costate variable to formulate the state equation as the constraint condition. We also pick up constraint conditions simultaneously imposed on the system. And we then define the Hamiltonian which describes development of the system in time consistently with these constraints. Next, we define and calculate the Dirac bracket. This is an extended form of the Poisson bracket in a way that the Dirac bracket is consistent with the constraints. Finally, we write down the Hamilton-Jacobi equation of the optimal control system as the constrained dynamical system.

### 2.1 An optimal control system and the constraint conditions

We define below the system and the control specification. Let us take $\vec{x} \in R^{n}$ as the state variable and $\vec{u} \in R^{m}$ as the control variable. The system is then described by the following state equation,

$$
\begin{equation*}
\dot{\vec{x}}=\vec{f}(\vec{x}, \vec{u}) \tag{1}
\end{equation*}
$$

The control specification imposed on the system is described by the following Lagrangian,

$$
\begin{equation*}
L(\vec{x}, \vec{u}) . \tag{2}
\end{equation*}
$$

Feedback is optimized such that the following time integral takes its minimum value,

$$
\begin{equation*}
\delta P I \equiv \delta\left\{\Phi\left(\vec{x}_{2}\right)+\int_{t_{1}}^{t_{2}} d t L(\vec{x}, \vec{u})\right\}=0 \tag{3}
\end{equation*}
$$

In the above equations, the function $\vec{f}$ is in general an arbitrarily nonlinear function and the Lagrangian $L$ is not restricted to quadratic form and takes an arbitrary form. The function $\Phi$ has also general form and this represents a cost of the terminal state $\vec{x}_{2}$. The following discussion holds even when the functions $\vec{f}$ and $L$ depend explicitly on time.

We use the costate variable $\vec{\lambda}$ with the same dimensionality as the state variable to define an extended Lagrangian,

$$
\begin{equation*}
L^{\prime}(\vec{x}, \vec{u}, \vec{\lambda} ; \dot{\vec{x}}) \equiv L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot(\vec{f}(\vec{x}, \vec{u})-\dot{\vec{x}}) \tag{4}
\end{equation*}
$$

where $\overleftarrow{\lambda} \equiv^{t} \vec{\lambda}$ is the transposed vector of $\vec{\lambda}$. Using this, only one condition,

$$
\begin{equation*}
\delta P I^{\prime} \equiv \delta\left\{\Phi\left(\vec{x}_{2}\right)+\int_{t_{1}}^{t_{2}} d t L^{\prime}(\vec{x}, \vec{u}, \vec{\lambda} ; \dot{\vec{x}})\right\}=0 \tag{5}
\end{equation*}
$$

describes the state equation (1) and the control specification (3) simultaneously. The state equation (1) is viewed as the constraint condition in this formulation. On the other hand, the constraint condition must also be satisfied along with wave motion associated with the optimal path. Therefore we must pick up other possible constraint conditions and need to check that they are consistent among them. Starting with calculations of canonical momenta, we find out all the constraint conditions in the following.

Regarding the optimal control system as the dynamical system described by the extended Lagrangian $L^{\prime}$, we can define canonical momenta by,

$$
\begin{equation*}
\vec{p}_{z}=\frac{\partial L^{\prime}}{\partial \dot{\vec{z}}}, \quad \quad z=x, u, \lambda \tag{6}
\end{equation*}
$$

The following three kinds of constraints are then introduced,

$$
\begin{align*}
& \vec{\phi}_{x} \equiv \vec{p}_{x}+\vec{\lambda} \approx 0  \tag{7}\\
& \vec{\phi}_{u} \equiv \vec{p}_{u} \approx 0  \tag{8}\\
& \vec{\phi}_{\lambda} \equiv \vec{p}_{\lambda} \approx 0 \tag{9}
\end{align*}
$$

where the notation " $\approx$ " denotes weak equality: these hold only over the subspace of the motion.

Taking into account that we have the constraints of Eqs. (7), (8) and (9), development of the system in time is determined by the following Hamiltonian,

$$
H\left(\vec{x}, \vec{u}, \vec{\lambda} ; \vec{p}_{x}, \vec{p}_{u}, \vec{p}_{\lambda}\right)
$$

$$
\begin{align*}
& \equiv \sum_{z=x, u, \lambda} \stackrel{\overleftarrow{p}}{z} \cdot \vec{z}-L^{\prime}(\vec{x}, \vec{u}, \vec{\lambda} ; \dot{\vec{x}}) \\
&+\sum_{z=x, u, \lambda} \overleftarrow{\mu}_{z} \cdot \vec{\phi}_{z} \tag{10}
\end{align*}
$$

In the above Eq.(10), $\vec{\mu}_{x}, \vec{\mu}_{u}$ and $\vec{\mu}_{\lambda}$ are Lagrange multipliers, which are determined so as to guarantee that the constraint conditions (7), (8) and (9) hold at any time. The time development of any dynamical variable $\omega$ is determined by $H$ as follows,

$$
\begin{equation*}
\dot{\omega}=\{\omega, H\}+\frac{\partial \omega}{\partial t} \tag{11}
\end{equation*}
$$

The first term in the above equation is the Poisson bracket between the dynamical variable $\omega$ and the Hamiltonian $H$. The Poisson bracket between any dynamical variables $\omega$ and $\sigma$ is defined as follows,

$$
\begin{equation*}
\{\omega, \sigma\} \equiv \sum_{z=x, u, \lambda}\left(\frac{\partial \omega}{\partial \overleftarrow{z}} \cdot \frac{\partial \sigma}{\partial \vec{p}_{z}}-\frac{\partial \sigma}{\partial \overleftarrow{z}_{z}} \cdot \frac{\partial \omega}{\partial \vec{p}_{z}}\right) \tag{12}
\end{equation*}
$$

After calculating time development of the constraint conditions (7), (8) and (9) according to Eqs.(11) and (12), we find further possible conditions which guarantee that these constraint conditions including the further conditions continue to hold at any time. First, we calculate the Hamiltonian (10) according to the definition of the canonical momenta (6) as follows,

$$
\begin{array}{r}
H\left(\vec{x}, \vec{u}, \vec{\lambda} ; \vec{p}_{x}, \vec{p}_{u}, \vec{p}_{\lambda}\right)=-L(\vec{x}, \vec{u})-\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u}) \\
+\sum_{z=x, u, \lambda} \overleftarrow{\mu}_{z} \cdot \vec{\phi}_{z} \tag{13}
\end{array}
$$

The time derivative of the constraint conditions (7) is calculated as,

$$
\begin{align*}
\dot{\phi}_{x_{i}}= & \left\{\phi_{x_{i}}, H\right\} \\
= & \left\{\phi_{x_{i}},-L(\vec{x}, \vec{u})-\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\right\} \\
& +\sum_{z=x, u, \lambda}\left\{\phi_{x_{i}}, \overleftarrow{\mu}_{z}\right\} \cdot \vec{\phi}_{z}+\sum_{z=x, u, \lambda} \overleftarrow{\mu}_{z} \cdot\left\{\phi_{x_{i}}, \vec{\phi}_{z}\right\} \\
\approx & \left\{\phi_{x_{i}},-L(\vec{x}, \vec{u})-\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\right\} \\
& +\sum_{z=x, u, \lambda} \overleftarrow{\mu}_{z} \cdot\left\{\phi_{x_{i}}, \vec{\phi}_{z}\right\} \\
= & \frac{\partial\{L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\}}{\partial x_{i}}+\mu_{\lambda_{i}} \approx 0 \tag{14}
\end{align*}
$$

which holds when we set one of the Lagrange multipliers $\vec{\mu}_{\lambda}$ as follows,

$$
\begin{equation*}
\mu_{\lambda_{i}} \approx-\frac{\partial\{L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\}}{\partial x_{i}} \tag{15}
\end{equation*}
$$

Next, the calculation of the time derivative of the constraint conditions (8) leads to the following,

$$
\begin{aligned}
\dot{\phi}_{u_{\alpha}} & =\left\{\phi_{u_{\alpha}}, H\right\} \\
& \approx\left\{\phi_{u_{\alpha}},-L(\vec{x}, \vec{u})-\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\right\}
\end{aligned}
$$

$$
\begin{align*}
& \quad+\sum_{z=x, u, \lambda} \overleftarrow{\mu}_{z} \cdot\left\{\phi_{u_{\alpha}}, \vec{\phi}_{z}\right\} \\
& =\frac{\partial\{L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\}}{\partial u_{\alpha}} \tag{16}
\end{align*}
$$

The above condition must be counted as a new constraint condition, because the condition (16) does not contain any Lagrange multiplier and any choice of the Lagrange multipliers $\vec{\mu}_{z}(z=x, u, \lambda)$ cannot guarantee the constraint condition (16). And a new constraint condition is the following,

$$
\begin{equation*}
\vec{\phi}_{H} \equiv \frac{\partial\{L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\}}{\partial \vec{u}} \approx 0 \tag{17}
\end{equation*}
$$

From the last constraints (9), we have,

$$
\begin{aligned}
& \dot{\phi}_{\lambda_{i}}=\left\{\phi_{\lambda_{i}}, H\right\} \\
& \approx\left\{\phi_{\lambda_{i}},-L(\vec{x}, \vec{u})-\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\right\} \\
&+\sum_{z=x, u, \lambda} \overleftarrow{\mu}_{z} \cdot\left\{\phi_{\lambda_{i}}, \vec{\phi}_{z}\right\}
\end{aligned}
$$

$$
\begin{equation*}
=f_{i}(\vec{x}, \vec{u})-\mu_{x_{i}} \approx 0 \tag{18}
\end{equation*}
$$

This holds when we set the following Lagrange multiplier,

$$
\begin{equation*}
\mu_{x_{i}} \approx f_{i}(\vec{x}, \vec{u}) \tag{19}
\end{equation*}
$$

The time derivative of the above new constraint condition (17) must also vanish,

$$
\begin{align*}
& \dot{\phi}_{H_{\alpha}}=\left\{\phi_{H_{\alpha}}, H\right\} \\
& \approx\left\{\phi_{H_{\alpha}},-L(\vec{x}, \vec{u})-\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\right\} \\
&+\sum_{z=x, u, \lambda} \overleftarrow{\mu}_{z} \cdot\left\{\phi_{H_{\alpha}}, \vec{\phi}_{z}\right\} \\
&= \mu_{x_{\underline{i}}} \frac{\partial^{2}\{L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\}}{\partial x_{\underline{i}} \partial u_{\alpha}} \\
&+\mu_{\underline{u_{\underline{\beta}}}} \frac{\partial^{2}\{L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\}}{\partial u_{\underline{\beta}} \partial u_{\alpha}} \\
& \quad+\mu_{\lambda_{\underline{i}}} \frac{\partial f_{\underline{i}}(\vec{x}, \vec{u})}{\partial u_{\alpha}} \approx 0 .(20) \tag{20}
\end{align*}
$$

(In the above equation (20), summing up over the range of indices is understood when the repetition of indices with underline arises, and we have $A_{\underline{\underline{i}}} B_{\underline{i}} \equiv \sum_{i=1}^{n} A_{i} B_{i}$ and $C_{\underline{\alpha}} D_{\underline{\alpha}} \equiv \sum_{\alpha=1}^{m} C_{\alpha} D_{\alpha}$.) Therefore when the following matrix with dimension $m$,

$$
\begin{equation*}
b_{\alpha, \beta} \equiv \frac{\partial^{2}\{L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\}}{\partial u_{\alpha} \partial u_{\beta}} \tag{21}
\end{equation*}
$$

is regular, we will use $\vec{\mu}_{x}$ calculated by Eq.(19) and $\vec{\mu}_{\lambda}$ calculated by Eq.(15) to get the following solution,

$$
\begin{align*}
\mu_{u_{\alpha}} & \approx-\left(b^{-1}\right)_{\alpha, \underline{\beta}}\left[f _ { \underline { i } } \left(\vec{x}, \vec{u} \frac{\partial^{2}\{L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\}}{\partial x_{\underline{i}} \partial u_{\underline{\beta}}}\right.\right. \\
& \left.-\frac{\partial\{L(\vec{x}, \vec{u})+\overleftarrow{\lambda} \cdot \vec{f}(\vec{x}, \vec{u})\}}{\partial x_{\underline{i}}} \frac{\partial f_{\underline{i}}(\vec{x}, \vec{u})}{\partial u_{\underline{\beta}}}\right] \tag{22}
\end{align*}
$$

In this paper we assume that the matrix $b$ is regular. The
regularity usually holds when the Lagrangian $L$ contains the control variable $\vec{u}$ in quadratic form. A study on systems where this matrix $b$ is not regular will be reported elsewhere. From the above equations, we determined the coefficients of $\vec{\mu}_{x}, \vec{\mu}_{u}$ and $\vec{\mu}_{\lambda}$ by Eqs.(19), (22) and (15), respectively. The new constraint condition (17) which is not expressed by a linear combination of the constraint conditions of Eqs.(7), (8) and (9) was added. This new constraint condition (17) expresses the optimality condition of control. No more constraint condition arises when we take time development of these four kind of constraint conditions. These four kinds of constraints close among themselves. The meaning is that the time derivative of any one $\dot{\phi}_{I}$ of these four kinds of constraint conditions is expressed as a linear combination $\dot{\phi}_{I} \sim \sum_{K} \phi_{K}$ of these four kinds of constraints. We can easily see from this formula that $\dot{\phi}_{I}=0$ if $\forall \phi_{K}=0$. And because $\ddot{\phi}_{I}$ is a linear combination $\sim \sum_{K} \dot{\phi}_{K}$, we also see that $\ddot{\phi}_{I}=0$ when $\phi_{K}=0$. We have at last that $\phi_{I}=0$ at any time with no additional constraint conditions.

## 2. 2 Dirac brackets

According to these four constraint conditions (7), (8), (9) and (17), let us define the Dirac bracket. the extended form of the Poisson bracket, Between any pair of dynamical variables, the Dirac bracket is defined by,

$$
\begin{equation*}
\{\omega, \sigma\}_{D B} \equiv\{\omega, \sigma\}-\left\{\omega, \phi_{\underline{I}}\right\}\left(K^{-1}\right)_{\underline{I}, \underline{J}}\left\{\phi_{\underline{J}}, \sigma\right\} . \tag{23}
\end{equation*}
$$

In this equation,

$$
\left(\phi_{I}\right) \equiv\left(\begin{array}{c}
\vec{\phi}_{x}  \tag{24}\\
\vec{\phi}_{u} \\
\vec{\phi}_{\lambda} \\
\vec{\phi}_{H}
\end{array}\right)
$$

and $K^{-1}$ is the inverse of $K$, each element $K_{I, J}$ of which is the Poisson bracket $\left\{\phi_{I}, \phi_{J}\right\}$ between any pair $\phi_{I}$ and $\phi_{J}$ of the four kinds of constraint conditions. As calculated in Appendix A, the matrix $K$ is calculated as follows,

$$
K \equiv\left(\left\{\phi_{I}, \phi_{J}\right\}\right)=\left(\begin{array}{cccc}
0 & 0 & I_{n} & -a  \tag{25}\\
0 & 0 & 0 & -b \\
-I_{n} & 0 & 0 & -c \\
{ }^{t} a & { }^{t} b & { }^{t} c & 0
\end{array}\right)
$$

where $I_{n}$ is the identity matrix of the order $n, a$ is the $n \times m$ matrix defined by,

$$
\begin{equation*}
a_{i, \beta} \equiv \frac{\partial^{2}(L+\overleftarrow{\lambda} \cdot \vec{f})}{\partial x_{i} \partial u_{\beta}} \tag{26}
\end{equation*}
$$

$b$ is the $m \times m$ matrix defined by Eq.(21) and $c$ is the $n \times m$ matrix defined by,

$$
\begin{equation*}
c_{i, \beta} \equiv \frac{\partial f_{i}}{\partial u_{\beta}} \tag{27}
\end{equation*}
$$

The Dirac bracket between any dynamical variable $\omega$ and any one of the constraint conditions $\phi_{K}$ vanishes as calculated in the following,

$$
\begin{align*}
\left\{\omega, \phi_{K}\right\}_{D B} & \equiv\left\{\omega, \phi_{K}\right\}-\left\{\omega, \phi_{\underline{I}}\right\}\left(K^{-1}\right)_{\underline{I}, \underline{J}}\left\{\phi_{\underline{J}}, \phi_{K}\right\} \\
& =\left\{\omega, \phi_{K}\right\}-\left\{\omega, \phi_{\underline{I}}\right\}\left(K^{-1}\right)_{\underline{I}, \underline{J}} K_{\underline{J}, K}=0 . \tag{28}
\end{align*}
$$

Each constraint condition $\phi_{K}$ is thus regarded to be identically zero in the Dirac bracket formulation. The inverse $K^{-1}$ is given in Appendix B. The inverse to be calculated is only $b^{-1}$. The matrix $b$ is given by Eq.(21) and has the dimensionality $m$ of that of the control variables. Irrespecive of the dimensionality $n$ of the state variable, usually $m<n$ or $m \ll n$ and a calculational burden of the inverse matrix $K^{-1}$ is small.
Nonvanishing Dirac brackets are given as follows,

$$
\begin{align*}
& \left\{x_{i}, p_{x_{j}}\right\}_{D B}=\delta_{i, j},  \tag{29}\\
& \left\{x_{i}, u_{\beta}\right\}_{D B}=\left(c b^{-1}\right)_{i, \beta},  \tag{30}\\
& \left\{x_{i}, \lambda_{j}\right\}_{D B}=-\delta_{i, j},  \tag{31}\\
& \left\{p_{x_{i}}, u_{\beta}\right\}=\left(a b^{-1}\right)_{i, \beta},  \tag{32}\\
& \left\{u_{\alpha}, u_{\beta}\right\}_{D B}=\left(\left(^{t} b\right)^{-1}\left(^{t} c a-^{t} a c\right) b^{-1}\right)_{\alpha, \beta},  \tag{33}\\
& \left\{u_{\alpha}, \lambda_{j}\right\}_{D B}=\left(\left(^{t} b\right)^{-1}{ }^{t} a\right)_{\alpha, j} . \tag{34}
\end{align*}
$$

We must note that the Dirac bracket (33) among the component $u_{\alpha}$ and $u_{\beta}$ of the control variable does not vanish, although the corresponding Poisson bracket is obviously zero. On the contrary, the Dirac bracket among the component $x_{i}$ and $x_{j}$ of the state variable vanishes: $\left\{x_{i}, x_{j}\right\}_{D B}=0$. In Sec.3.1 below, we give a linear operator representation $\hat{\xi}$ of each canonical variable $\xi$. And we set a commutation relation $[\hat{\xi}, \hat{\eta}] \equiv \hat{\xi} \hat{\eta}-\hat{\eta} \hat{\xi}$ corresponding to the Dirac bracket $\{\xi, \eta\}_{D B}$ between a canonical variable $\xi$ and another canonical variable $\eta$ in a way consistent with the algebraic structure among the Dirac brackets (29) to (34) above. The vanishing Dirac bracket $\left\{x_{i}, x_{j}\right\}_{D B}=0$ then corresponds to the vanishing commutation relation $\left[\hat{x}_{i}, \hat{x}_{j}\right]=0$. That $\hat{x}_{i}$ and $\hat{x}_{j}$ commutes, $\hat{x}_{i} \hat{x}_{j}=\hat{x}_{j} \hat{x}_{i}$, means that the state variable $\vec{x}$ remains the classical variable. And this feature of the commutation relations enables us to represent an operator $\hat{\xi}$ as a linear map $\hat{a}: f \mapsto g$ on functional space of state functions $f=f(\vec{x}), g=g(\vec{x})$.

The equation of motion represented by the Dirac bracket is given by,

$$
\begin{equation*}
\dot{\omega}=\{\omega, H\}_{D B}+\frac{\partial \omega}{\partial t} . \tag{35}
\end{equation*}
$$

For the optimal control system as the constrained dynamical system, let us define a function $S(\vec{x}, \vec{u}, \vec{\lambda} ; t)$ of
the canonical coordinates $(\vec{x}, \vec{u}, \vec{\lambda})$ as that satisfying the following Hamilton-Jacobi equation,

$$
\begin{equation*}
H\left(\vec{x}, \vec{u}, \vec{\lambda} ; \vec{p}_{x}, \vec{p}_{u}, \vec{p}_{\lambda}\right)+\frac{\partial S(\vec{x}, \vec{u}, \vec{\lambda} ; t)}{\partial t}=0, \tag{36}
\end{equation*}
$$

where

$$
\begin{equation*}
\vec{p}_{z}=\frac{\partial S(\vec{x}, \vec{u}, \vec{\lambda} ; t)}{\partial \vec{z}}, \quad z=x, u, \lambda . \tag{37}
\end{equation*}
$$

We write down explicitly the Hamilton-Jacobi equation for the following affine system. This system is practically important and is defined by the following state equations and a Lagrangian,

$$
\begin{align*}
& f_{i}(\vec{x}, \vec{u})=g_{i, \underline{\alpha}}(\vec{x}) u_{\underline{\alpha}}+F_{i}(\vec{x}),  \tag{38}\\
& L(\vec{x}, \vec{u})=u_{\underline{\alpha}} R_{\underline{\alpha}, \underline{\beta}} u_{\underline{\beta}}+V_{\text {cost }}(\vec{x}) . \tag{39}
\end{align*}
$$

Use of the Dirac bracket allows us to set $\vec{\phi}_{x}=\vec{\phi}_{u}=\vec{\phi}_{\lambda}=$ 0 and by setting the last term of $H, \sum_{z=x, u, \lambda} \overleftarrow{\mu}_{z} \cdot \vec{\phi}_{z}$ to be zero we have,

$$
\begin{align*}
H & =-L-\overleftarrow{\lambda} \cdot \vec{f} \\
& =-u_{\underline{\alpha}} R_{\underline{\alpha}, \underline{\beta}} u_{\underline{\beta}}-V_{\text {cost }}-\lambda_{\underline{i}}\left(g_{\underline{i}, \underline{\alpha}} u_{\underline{\alpha}}+F_{\underline{i}}\right) . \tag{40}
\end{align*}
$$

From the optimality condition $\phi_{H}=0$, Eq.(17), we have the relation between $u_{\alpha}$ and $\lambda_{i}$ as

$$
\begin{equation*}
u_{\alpha}=-\left(\bar{R}^{-1}\right)_{\alpha, \underline{\alpha}} \lambda_{\underline{i}} g_{i, \underline{\alpha}}, \tag{41}
\end{equation*}
$$

where $\bar{R}$ is the inverse matrix of $R+{ }^{t} R$. Expressing $\vec{\lambda}$ according to the constraint condition $\vec{\phi}_{x}=0$, Eq.(7),

$$
\begin{equation*}
\vec{\lambda}_{x}=-\vec{p}_{x}=-\frac{\partial S}{\partial \vec{x}}, \tag{42}
\end{equation*}
$$

leads to the following Hamilton-Jacobi equation,

$$
\begin{aligned}
& \frac{\partial S}{\partial t}+\frac{1}{2}\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}} g_{\underline{i}, \underline{\alpha}} g_{\underline{j}, \underline{\beta}} \frac{\partial S}{\partial x_{\underline{i}}} \frac{\partial S}{\partial x_{\underline{j}}} \\
&-V_{\text {cost }}+F_{\underline{j}} \frac{\partial S}{\partial x_{\underline{j}}}=0 .(43)
\end{aligned}
$$

## 3. Formulation of the optimal control systems by linear operators

Once we have obtained the Dirac bracket we can handle the optimal control system without paying any attention to the constraints. In this section linear operators which are in one-to-one correspondence to the canonical variables of the optimal control system are defined. The Dirac bracket $\{\omega, \sigma\}_{D B}$ between any pair of dynamical variables $\omega$ and $\sigma$, is then replaced with a commutation realtion $[\hat{\omega}, \hat{\sigma}] \equiv \hat{\omega} \hat{\sigma}-\hat{\omega} \hat{\sigma}$ between corresponding operators $\hat{\omega}$ and $\hat{\sigma}$. And these linear operators are determined by a requirement that the commutation relations are to satisfy the algebraic structure that is found in the Dirac brackets between canonical variables ${ }^{8)}$. The canonical variables are classical variables and the commutation relations between the classical variables are obviously zero, $[\omega, \sigma]=\omega \sigma-\sigma \omega=0$. However, the canonical variables
represented as linear operators have in general nonvanishing commutation relations and to represent an amount of the nonvanishing value new real positive constant $H_{R}$ is introduced. To the control variable corresponds a control variable operator, and to the costate variable corresponds a costate variable operator, respectively. We find from properties of the algebraic structure, that these linear operators can be represented as those which map one $f=f(\vec{x})$ to another $g=g(\vec{x})$ of the state functions of the state variable. Especially for the affine system, these linear operators are explicitly given in a combined form of the state variable and its partial differential operation. A Hamiltonian operator that corresponds to the Hamiltonian is also defined and calculated. And using this Hamiltonian operator, we set a linear wave equation imposed on a wave function which is a complex-valued state function. We clarify that in sufficiently weak wave action the linear wave equation approximates the original optimal control system. In other words, the phase function of the wave function satisfies an equation in a form of the HamiltonJacobi equation with an additional term representing the wave action. And the generalized Hamilton-Jacobi equation approaches the conventional Hamilton-Jacobi equation when we set this additional term sufficiently small.

## 3. 1 Definitions of the linear operators satisfying the commutation relations

We set a linear operator $\hat{\omega}$ corresponding to a dynamical variable $\omega$. The canonical variable $\omega$ is the classical variable. The meaning is that this is not an operator. The condition of the correspondence is that the algebraic structure of the Dirac brackets Eqs.(29) to (34) calculated before is also satisfied even when we replace the Dirac bracket $\{\omega, \sigma\}_{D B}$ with the following commutation relation among the corresponding linear operators $\hat{\omega}$ and $\hat{\sigma}$,

$$
\begin{equation*}
i H_{R}\{\omega, \sigma\}_{D B} \rightarrow[\hat{\omega}, \hat{\sigma}] \equiv \hat{\omega} \hat{\sigma}-\hat{\sigma} \hat{\omega} \tag{44}
\end{equation*}
$$

In this equation (44) a parameter $H_{R}$ is a real and positive constant which is appropriately set by control designers. As explained in the following, this new control constant is a "unit" to nondimensionalize the phase function of the wave associated with the optimal control system and it represents strength of the wave action. In the above Eq.(44), $i \equiv \sqrt{-1}$ is the imaginary unit. The algebraic structure represented by the Dirac bracket in section 2.2 is replaced with the following formulas,

$$
\begin{align*}
& {\left[x_{i}, \hat{u}_{\beta}\right]=i H_{R} \overline{\left(c b^{-1}\right)_{i, \beta}}}  \tag{45}\\
& {\left[x_{i}, \hat{\lambda}_{j}\right]=-i H_{R} \delta_{i, j}} \tag{46}
\end{align*}
$$

$$
\begin{align*}
& {\left[\hat{u}_{\alpha}, \hat{u}_{\beta}\right]=i H_{R} \overline{\left(b^{-1}\left(t c a-{ }^{t} a c\right) b^{-1}\right)_{\alpha, \beta}}}  \tag{47}\\
& {\left[\hat{u}_{\alpha}, \hat{\lambda}_{j}\right]=i H_{R} \overline{\left(b^{-1 t} a\right)_{\alpha, j}}} \tag{48}
\end{align*}
$$

We call $\hat{\vec{u}}$ control variable operator and $\hat{\vec{\lambda}}$ costate variable operator. We set $\wedge$ over these operators to clarify differences between these operators and classical control variable $\vec{u}$ and costate variable $\vec{\lambda}$. The overbar appearing in the r.h.s. of Eq. (45) to (48) above is defined as,

$$
\begin{equation*}
\overline{\hat{\omega} \hat{\sigma}} \equiv \frac{\hat{\omega} \hat{\sigma}+\hat{\sigma} \hat{\omega}}{2} \tag{49}
\end{equation*}
$$

which is a symmetrization of an arbitrary product of linear operators. We note again that the components of the state variable commute each other. By this nontrivial fact we can represent the above linear operators as linear maps on the space of the state functions. In the above equations, the commutation relations corresponding to the Dirac brackets (29) and (32) including $\vec{p}_{x}$ give the same information as those given by Eqs.(46) and (48), and these are omitted.

The linear operators are completely determined by the above commutation relations (45) to (48). In the following we explicitly write down the linear operator representations for the affine system of Eqs.(38) and (39). The $a, b$ and $c$ matrices of Eqs.(26), (21) and (27) are calculated as follows,

$$
\begin{align*}
a_{i, \beta} & =\lambda_{\underline{j}} \frac{\partial g_{\underline{j}, \beta}(\vec{x})}{\partial x_{i}}  \tag{50}\\
b_{\alpha, \beta} & =R_{\alpha, \beta}+R_{\beta, \alpha}  \tag{51}\\
c_{i, \beta} & =g_{i, \beta}(\vec{x}) \tag{52}
\end{align*}
$$

respectively. After the symmetrization (49), substituting these matrices into r.h.s. of Eqs.(45) to (48) leads to the linear operators as partial derivatives $\frac{\partial}{\partial \vec{x}}$ multiplied by function coefficients as follows,

$$
\begin{align*}
& \hat{u}_{\alpha}=-\left(\bar{R}^{-1}\right)_{\alpha, \underline{\beta}} i H_{R} \nabla^{g_{j} \underline{\underline{\beta}}, \underline{j}},  \tag{53}\\
& \hat{\lambda}_{i}=i H_{R} \frac{\partial}{\partial x_{i}} . \tag{54}
\end{align*}
$$

In the above equation we define for any function $h(\vec{x}, t)$ of $\vec{x}$ and $t$,

$$
\begin{equation*}
\nabla_{i}^{h} \equiv h \frac{\partial}{\partial x_{i}}+\frac{1}{2} \frac{\partial h}{\partial x_{i}} \tag{55}
\end{equation*}
$$

The Hamiltonian operator $\hat{H}$ after the symmetrization is calculated by the following formula,

$$
\begin{align*}
\hat{H} & =-\overline{L(\vec{x}, \hat{\vec{u}})-\hat{\overleftarrow{\lambda}} \cdot \vec{f}(\vec{x}, \hat{\vec{u}})} \\
& =-\overline{\hat{u}_{\underline{\alpha}} R_{\underline{\alpha}, \underline{\beta}} \hat{u}_{\underline{\beta}}}-V_{\text {cost }}(\vec{x})-\overline{\hat{\lambda}_{\underline{i}} \hat{u}_{\underline{\alpha}} g_{\underline{i}, \underline{\alpha}}(\vec{x})}-\overline{\hat{\lambda}_{\underline{j}} F_{\underline{j}}(\vec{x})} \\
& =-\frac{H_{R}^{2}}{2}\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}} \overline{\nabla^{g_{i, \underline{\alpha}}^{\underline{\alpha}}} \nabla^{g_{j}-\mathcal{T}_{\underline{j}}}}-V_{\text {cost }}-i H_{R} \nabla_{\underline{j}}^{F_{j}} . \tag{56}
\end{align*}
$$

## 3. 2 The linear wave equation and the generalized Hamilton-Jacobi equation

We use the Hamiltonian operator to set up a linear wave equation imposed on a complex-valued wave function $\psi\left(\vec{x}, t ; H_{R}\right)$, which leads to a generalized equation of the Hamilton-Jacobi equation. A linear wave equation is given by the following,

$$
\begin{equation*}
i H_{R} \frac{\partial \psi\left(\vec{x}, t ; H_{R}\right)}{\partial t}=\hat{H} \psi\left(\vec{x}, t ; H_{R}\right) . \tag{57}
\end{equation*}
$$

In this equation, we explicitly show dependence of the wave function $\psi$ on the parameter $H_{R}$. We note that this equation (57) is a linear partial differential equation both in time and space coordinates. This reflects the fact that waves are superposable. Next we decompose the wave function to the following combination of an absolute value function and a phase function,

$$
\begin{equation*}
\psi\left(\vec{x}, t ; H_{R}\right)=R^{q}\left(\vec{x}, t ; H_{R}\right) \exp \left(\frac{i}{H_{R}} S^{q}\left(\vec{x}, t ; H_{R}\right)\right), \tag{58}
\end{equation*}
$$

where we set $H_{R}$ as a unit of the phase function. We explicitly write down dependence on $H_{R}$ of the absolute value function $R^{q}$ and the dimensional phase function $S^{q}$. We note that in the above equation, the wave function depends explicitly on time and so the wave equation contains time partial derivative. Substituting the expression (58) into the wave equation (57) and taking the real part projection lead to an generalization of the HamiltonJacobi equation. For the affine system of Eqs.(38) and (39), as will be shown in Appendix C, the phase function $S^{q}\left(\vec{x}, t, H_{R}\right)$ satisfies the following partial differential equation,

$$
\begin{align*}
& \frac{\partial S^{q}}{\partial t}-V^{q}+\frac{\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}}}{2} g_{\underline{i}, \underline{\alpha}} g_{\underline{j}, \underline{\beta}} \frac{\partial S^{q}}{\partial x_{\underline{i}}} \frac{\partial S^{q}}{\partial x_{\underline{j}}} \\
&-V_{\text {cost }}+F_{\underline{j}} \frac{\partial S^{q}}{\partial x_{\underline{j}}}=0 . \tag{59}
\end{align*}
$$

Except the second term in the l.h.s., the above equation is nothing but the Hamilton-Jacobi equation (43). This term $V^{q}$ is given by,

$$
\begin{equation*}
V^{q}\left(\vec{x}, t ; H_{R}\right)=\frac{H_{R}^{2}}{2}\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}} \frac{w_{\underline{\alpha}, \boldsymbol{\beta}}^{q}\left(\vec{x}, t ; H_{R}\right)}{R^{q}\left(\vec{x}, t ; H_{R}\right)} . \tag{60}
\end{equation*}
$$

We call this an additional cost, because this $V^{q}$ contributes to the equation in a form added to the conventional cost function $V_{\text {cost }}$. This additional cost $V^{q}$ represents wave strength acted on the optimal path. In the above equation (60), the denominator is calculated as,

$$
\begin{aligned}
w_{\alpha, \beta}^{q} & \equiv g_{\underline{i}, \alpha} g_{\underline{j}, \beta} \frac{\partial^{2} R^{q}}{\partial x_{\underline{i}} \partial x_{\underline{j}}} \\
& +\frac{1}{2}\left(g_{\underline{i}, \alpha} \frac{\partial g_{\underline{j}, \beta}}{\partial x_{\underline{i}}}+g_{\underline{j}, \beta} \frac{\partial g_{\underline{i}, \alpha}}{\partial x_{\underline{i}}}\right) \frac{\partial R^{q}}{\partial x_{\underline{j}}}
\end{aligned}
$$

$$
\begin{gather*}
+\frac{1}{2}\left(g_{\underline{i}, \alpha} \frac{\partial g_{\underline{j}, \beta}}{\partial x_{\underline{j}}}+g_{\underline{j}, \beta} \frac{\partial g_{\underline{i}, \alpha}}{\partial x_{\underline{j}}}\right) \frac{\partial R^{q}}{\partial x_{\underline{i}}} \\
+\frac{1}{4}\left(g_{\underline{i}, \alpha} \frac{\partial^{2} g_{\underline{j}, \beta}}{\partial x_{\underline{i}} \partial x_{\underline{j}}}+\frac{\partial g_{\underline{i}, \alpha}}{\partial x_{\underline{i}}} \frac{\partial g_{\underline{j}, \beta}}{\partial x_{\underline{j}}}+g_{\underline{j}, \beta} \frac{\partial^{2} g_{\underline{i}, \alpha}}{\partial x_{\underline{i}} \partial x_{\underline{j}}}\right) R^{q} . \tag{61}
\end{gather*}
$$

This additional cost $V^{q}$ contains division by the absolute value function $R^{q}$ and has possibly a complicated dependence on $H_{R}$, the origin of which is the possibly complicated dependence of the wave function on $H_{R}$. However, an appropriate time boundary condition leads to convergence of $V^{q} \rightarrow 0$ in the order of $H_{R}^{2}$ in $H_{R} \rightarrow 0$. The proof needs detailed discussion on the boundary condition and will be reported elsewhere. We call Eq.(59) as a generalized Hamilton-Jacobi equation. When we set $H_{R}$ sufficiently small the generalized Hamilton-Jacobi equation (59) approximates the conventional Hamilton-Jacobi equation (43), because the order of the additional cost $V^{q}$ is $H_{R}^{2}$.

## 3. 3 A state feedback law

We show in the following how to calculate optimal state feedback by making use of a solution of the wave function. The state feedback is obtained from the optimality condition of control, Eq.(17). For the affine system this is nothing but Eq.(41). According to the constraint condition (42) between the canonical momentum of the state and costates variable, we attain,

$$
\begin{equation*}
u_{\alpha}=(\bar{R})_{\alpha, \underline{\alpha}} \frac{\partial S}{\partial x_{\underline{i}}} g_{\underline{i}, \underline{\alpha}} . \tag{62}
\end{equation*}
$$

In this equation, $S$ is a solution of the Hamilton-Jacobi equation (43) which we approximate by the phase function $S^{q}$ of the wave function $\psi$. Straightforward calculations of differentiations of the wave function leads to the following formula of the optimal feedback,

$$
\begin{align*}
u_{\alpha} & =\left(\bar{R}^{-1}\right)_{\alpha, \underline{\alpha}} \frac{\partial S^{q}}{\partial x_{\underline{\underline{i}}}} g_{\underline{i}, \underline{\alpha}} \\
& =\left(\bar{R}^{-1}\right)_{\alpha, \underline{,}} H_{R} \frac{\operatorname{Re} \psi \frac{\partial \operatorname{Im\psi }}{\partial x_{\underline{i}}}-\operatorname{Im} \psi \frac{\partial R e \psi}{\partial x_{\underline{i}}}}{(\operatorname{Re} \psi)^{2}+(\operatorname{Im} \psi)^{2}} g_{\underline{i}, \underline{\alpha}} . \tag{63}
\end{align*}
$$

In the r.h.s. of the above equation $H_{R}$ is multiplied because the phase function $S^{q}$ appears in the wave function in the form of $\frac{S^{q}}{H_{R}}$.

## 4. A numerical example

We show a numerical example of the optimal feedback control of a system containing nonlinearities both in the state equation and the control specification. After setting up the wave equation according to the method stated above, we show simulation studies. We will check that the Hamilton-Jacobi equation is satisfied when the new
designer's constant $H_{R}$ tends to zero. We will also examine behaviors of the additional costs. Our aim here is to present in a concreet way the new concept clearly. For this purpose we deal with a simple system with 1-input and 1-state. We will show systematically in forthcoming papers methods of calculations for general systems with multiple inputs and states.

## 4. 1 The linear wave equation and the optimal feedback

In the following, we will show an algorithm. We start with a definition of a system to be examined and then give calculations of feedback control laws.

A system is described by the following nonlinear state equation and the non-quadratic cost function,

$$
\begin{align*}
& g(x)=B, \quad F(x)=A x+\delta x^{3}  \tag{64}\\
& R=\frac{M}{2}, \quad V_{\operatorname{cost}}(x)=K\left(x^{2}+\beta x^{4}\right) \tag{65}
\end{align*}
$$

In the above equations, we set $B=1, A=1, \delta=0.1$, $M=1, K=0.5$ and $\beta=0.1$. When there is no $F(x)$ term in the above state equation ${ }^{10)}$, the state equation is nothing but a definition of a velocity $(\dot{x}=u)$ in classical mechanics of point particles and we can set up our linear wave equation entirely in the same manner of theories of a quantum wave associated with point particles. However, in control theories a time derivative of the state variable is calculated using a state function like the function $F(x)$ defined by Eq.(64). We thus no longer see the state equation as the definition of the velocity. Even for such system, we can set up a linear wave equation according to the method described in the foregoing sections.

The control variable operator, the costate variable operator and the Hamiltonian operator are calculated as follows, according to (53), (54) and (56),

$$
\begin{align*}
& \hat{u}=-i \frac{1}{M} H_{R} \frac{\partial}{\partial x}  \tag{66}\\
& \begin{aligned}
& \hat{\lambda}=i H_{R} \frac{\partial}{\partial x} \\
& \begin{aligned}
\hat{H} & =-\frac{H_{R}^{2}}{2 M} \frac{\partial^{2}}{\partial x^{2}}- \\
& -V_{\operatorname{cost}}(x) \\
& -i H_{R}\left\{F(x) \frac{\partial}{\partial x}+\frac{1}{2} \frac{\partial F(x)}{\partial x}\right\}
\end{aligned}
\end{aligned} \begin{aligned}
& \\
&
\end{aligned} \tag{67}
\end{align*}
$$

respectively. Using these we can set up the linear wave equation (57) as follows,

$$
\begin{align*}
i H_{R} \frac{\partial \psi(x ; t)}{\partial t} & =\left[-\frac{H_{R}^{2}}{2 M} \frac{\partial^{2}}{\partial x^{2}}-V_{\text {cost }}(x)\right. \\
- & \left.i H_{R}\left\{F(x) \frac{\partial}{\partial x}+\frac{1}{2} \frac{\partial F(x)}{\partial x}\right\}\right] \psi(x ; t) \tag{69}
\end{align*}
$$

According to Eqs.(59) and (41) we find that the generalized Hamilton-Jacobi equation and the optimal feedback


Fig. 3 Results of optimal control with $H_{R}=1.0$


Fig. 4 Results of optimal control with $H_{R}=0.5$


$$
\begin{aligned}
& 1 \text { : State Variable } \quad-3 \sim 2 \\
& 2 \text { : Manipulated Variable }-20 \sim 0 \\
& 3 \text { : Error of } \\
& \quad \text { Itamilton-Jacobi Equation } \\
& \qquad \begin{aligned}
-20 \sim 30[\%]
\end{aligned} \\
& \begin{aligned}
4 \text { : Error of generalized }
\end{aligned} \\
& \text { Hamilton-Jacobi Equation } \\
& \\
& \\
& -10 \sim 40[\%]
\end{aligned}
$$

Fig. 5 Results of optimal control with $H_{R}=0.1$
control are given as follows,

$$
\begin{align*}
& \begin{array}{l}
\frac{\partial S^{q}(x ; t)}{\partial t}-\frac{H_{R}^{2}}{2 M} \frac{1}{R^{q}(x ; t)} \frac{\partial^{2} R^{q}(x ; t)}{\partial x^{2}} \\
\quad+\frac{1}{2 M}\left(\frac{\partial S^{q}(x ; t)}{\partial x}\right)^{2}-V_{\cos t}(x) \\
\\
+F(x) \frac{\partial S^{q}(x ; t)}{\partial x}=0
\end{array} \\
& \begin{array}{l}
u=\frac{1}{M} \frac{\partial}{\partial x} S^{q}(x ; t)
\end{array}
\end{align*}
$$

respectively.

## 4. 2 Simulation

We will show simulation studies of feedback optimizations of the above nonilnear control system. We use a perturbation method typical to the quantum mechanical linear wave equation. And we mainly compare profiles of the additional cost $V^{q}$ by taking three values of the control constant $H_{R}$. We then make use of such examinations to clarify that the generalized Hamilton-


Fig. 6 Absolute value $R^{q}$, its 2nd derivative $\frac{\partial^{2} R^{q}}{\partial x^{2}}$ and their ratio with $H_{R}=1.0$


Fig. 7 Absolute value $R^{q}$, its 2 nd derivative $\frac{\partial^{2} R^{q}}{\partial x^{2}}$ and their ratio with $H_{R}=0.5$


Fig. 8 Absolute value $R^{q}$, its 2 nd derivative $\frac{\partial^{2} R^{q}}{\partial x^{2}}$ and their ratio with $H_{R}=0.1$


Fig. $9 \frac{\partial S^{q}}{\partial t}\left(\right.$ Time derivative of phase value $\left.S^{q}\right)$ evaluated at various $H_{R}$ 's

Jacobi equation approximates the conventional one when we set $H_{R}$ as small values. First, we show time developments of the control system. Setting the control constant $H_{R}=1.0,0.5$ and 0.1 , the corresponding trends are shown in Fig.3, Fig. 4 and Fig.5, respectively. In these figures, we show the trends of the state variables(No. 1 with the range of $-3 \sim 2$ ), the control variables(No. 2 with the range of $-20 \sim 0$ ), the errors of the Hamilton-Jacobi equations(No.3, with the range of $-20 \sim 30$ ) and those of the generalized Hamilton-Jacobi equations(No. 4 with the range of $-10 \sim 40$ ). We here calculate the error of the Hamilton-Jacobi equation as,

$$
\begin{array}{r}
{\left[\frac{1}{2 M}\left(\frac{\partial S^{q}(x ; t)}{\partial x}\right)^{2}-V_{\text {cost }}(x)+F(x) \frac{\partial S^{q}(x ; t)}{\partial x}\right]} \\
/\left(K x_{I C}^{2}\right)=0 \tag{72}
\end{array}
$$

In the above equation, we use the initial value $x_{I C}=1$ of the state variable to nondimensionalize the numerator. The error of the generalized Hamilton-Jacobi equation (70) is the l.h.s. of this equation, which is also nondimensionalized by $K x_{I C}^{2}$ as follows,

$$
\begin{align*}
{\left[\frac{\partial S^{q}(x ; t)}{\partial t}-\right.} & \frac{H_{R}^{2}}{2 M} \frac{1}{R^{q}(x, t)} \frac{\partial^{2} R^{q}(x, t)}{\partial x^{2}} \\
+\frac{1}{2 M} & \left(\frac{\partial S^{q}(x ; t)}{\partial x}\right)^{2}-V_{\cos t}(x) \\
& \left.+F(x) \frac{\partial S^{q}(x ; t)}{\partial x}\right] /\left(K x_{I C}^{2}\right)=0 \tag{73}
\end{align*}
$$

The errors of the generalized Hamilton-Jacobi equation near the initial time take large values ( $\sim 4 \%$ ). These large values suggest inadequate application of the perturbation theory to the initial large value of $x_{I C}=1$. However, these errors of the generalized Hamilton-Jacobi equation keep approximately zero except a neighborhood of the initial time. And these trends in time depend little on the control constant $H_{R}$. On the other hand, time trends of the errors of the Hamilton-Jacobi equation depend on the control constant $H_{R}$ and by taking the $H_{R}$ small these approach the error trends of the generalized Hamilton-Jacobi equation. In other words, solutions of the Hamilton-Jacobi equation can be approximated by those of the generalized Hamilton-Jacobi equation in the zero limit of the control constant $H_{R}$.

Next, we show spatial profiles of the additional cost $V^{q}(x ; t)$ of Eq.(70) and examine their dependence on $H_{R}$. The additional cost is calculated by the following formula,

$$
\begin{equation*}
V^{q}(x ; t) \equiv \frac{H_{R}^{2}}{2 M} \frac{1}{R^{q}} \frac{\partial^{2} R^{q}}{\partial x^{2}} \tag{74}
\end{equation*}
$$

In Fig.6, Fig. 7 and Fig.8, ratios of the 2nd derivative $\frac{\partial^{2}}{\partial x^{2}} R^{q}$ to the absolute value function $R^{q}$ (No. 1 in these Figures with the range $-1.2 \sim 0.8$ ) along with the 2 nd
derivative $\frac{\partial^{2}}{\partial x^{2}} R^{q}$ (No. 2 with the range $-0.8 \sim 1.2$ ) and the absolute value function $R^{q}$ (No. 3 with the range $0.9 \sim$ 1.4) are shown. The control constants are set again as $H_{R}=1.0,0.5$ and 0.1 for Fig.6, Fig. 7 and Fig.8, respectively. As a result of calculations, these profiles varies little in time. Comparison of these three Figures shows that the spatial profiles of the ratio of $\frac{\partial^{2} R^{q}(x ; t)}{\partial x^{2}}$ to $R^{q}(x ; t)$ have little dependence on $H_{R}$. This means that the additional cost calculated by Eq.(74) vary approximately in proportional to the square of $H_{R}$ and tends to zero in the zero limit of $H_{R}$. In these example calculations, the absolute value function $R^{q}(x ; t)$, the denominator of Eq.(74), has no zero point. We expect that the zero points of such denominator can be avoided by taking an appropriate time boundary condition. This point will be examined in detail in forthcoming papers. The first term of the generalized Hamilton-Jacobi equation (70), the time derivative of the phase function, $\frac{\partial S^{q}}{\partial t}$, has almost constant value over both time and state variables. As shown in Fig.9(with the range $-0.4 \sim 0.1$ ) these tend to zero in proportional to the square of $H_{R}$.

## 5. Summary

For the purpose of calculating optimal feedback control for a nonlinear system, we assumed that associated with the optimal path there is a linear superposable complexvalued wave. To represent strength of action of the wave, we introduced a constant $H_{R}$. Regarding an optimal control system as a dynamical system with constraints of state equations, we defined linear operators which are in one-to-one correspondence to canonical variables. These operators were arranged such that these satisfy a certain algebraic structure. This is the structure that appears in the algebra among the Dirac brackets of the canonical variables. Especially for an affine system, the linear operators were explicitly represented as combinations of partial differentials regarding to state variables. After a calculation of a Hamiltonian operator corresponding to the Hamiltonian function, we set up a linear wave equation for a wave function. We showed that a phase function of the wave function satisfies an equation. This equation is the same as Hamilton-Jacobi equation except that there arises an additional cost term representing the action of the wave. We called this equation the generalized Hamilton-Jacobi equation. Taking into account that the additional cost is proportional to $H_{R}^{2}$, we showed that the generalized Hamilton-Jacobi equation approaches the conventional Hamilton-Jacobi equation when we set $H_{R}$ as small as possible. We explained a method of calculat-
ing nonlinear optimal feedback control by a solution of the wave equation. To clarify new concepts and to show validity of our algorithm of the feedback calculation, we gave numerical simulation studies of a simple nonlinear system with 1 -input and 1 -state variable.

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## Appendix A. Calculations of the Poisson brackets between the constraint conditions

In evaluations of the Dirac bracket, we must calculate the Poisson brackets between the constraint conditions. These calculations are not restricted to affine sytems. The Poisson brackets between the canonical variables are calculated as follows,

$$
\begin{align*}
& \left\{x_{i}, p_{x_{j}}\right\}=\delta_{i, j},  \tag{A.1}\\
& \left\{u_{\alpha}, p_{u_{\beta}}\right\}=\delta_{\alpha, \beta},  \tag{A.2}\\
& \left\{\lambda_{i}, p_{\lambda_{j}}\right\}=\delta_{i, j} . \tag{A.3}
\end{align*}
$$

The Poisson brackets of the other combinations are zero. The following formulas will also be useful,

$$
\begin{equation*}
\left\{f(\vec{x}), p_{x_{j}}\right\}=\frac{\partial f(\vec{x})}{\partial x_{j}}, \tag{A.4}
\end{equation*}
$$

$$
\begin{align*}
\left\{f(\vec{u}), p_{u_{\beta}}\right\} & =\frac{\partial f(\vec{u})}{\partial u_{\beta}}  \tag{A.5}\\
\left\{f(\vec{\lambda}), p_{\lambda_{j}}\right\} & =\frac{\partial f(\vec{\lambda})}{\partial \lambda_{j}} \tag{A.6}
\end{align*}
$$

Using the above obtained formulas, we can calculate the Poisson brackets between the constraint conditions.
(1) The Poisson brackets including $\phi_{x}$

The Poisson bracket between $\phi_{x}$ and $\phi_{x}$ or that between $\phi_{x}$ and $\phi_{u}$ are not included in Eqs.(A.1), (A.2) and (A.3) and these vanish. For the combination of $\phi_{x}$ and $\phi_{\lambda}$, we see that,

$$
\begin{equation*}
\left\{\phi_{x_{i}}, \phi_{\lambda_{j}}\right\}=\delta_{i, j} \tag{A.7}
\end{equation*}
$$

by making use of Eqs.(A.3) or (A.6). And for the combination of $\phi_{x}$ and $\phi_{H}$, we see from (A.4) that,

$$
\begin{equation*}
\left\{\phi_{x_{i}}, \phi_{H_{\beta}}\right\}=\left\{p_{x_{i}}, \phi_{H_{\beta}}\right\}=-\frac{\phi_{H_{\beta}}}{\partial x_{i}} \tag{A.8}
\end{equation*}
$$

which is nothing but the minus sign of Eq.(26)
(2) The Poisson brackets including $\phi_{u}$

The Poisson bracket between $\phi_{u}$ and $\phi_{u}$ or that between $\phi_{u}$ and $\phi_{\lambda}$ are not included in (A.1), (A. 2) and (A.3) and these vanish. For the combination of $\phi_{u}$ and $\phi_{H}$, we use Eq.(A.5) to obtain,

$$
\begin{equation*}
\left\{\phi_{u_{\alpha}}, \phi_{H_{\beta}}\right\}=\left\{p_{u_{\alpha}}, \phi_{H_{\beta}}\right\}=-\frac{\phi_{H_{\beta}}}{\partial u_{\alpha}} \tag{A.9}
\end{equation*}
$$

which is nothing but the minus sign of Eq.(21).
(3) The Poisson bracktes including $\phi_{\lambda}$

For the combination of $\phi_{\lambda}$ and $\phi_{H}$, we use Eq.(A.6) to obtain,

$$
\begin{equation*}
\left\{\phi_{\lambda_{i}}, \phi_{H_{\beta}}\right\}=\left\{p_{\lambda_{i}}, \phi_{H_{\beta}}\right\}=-\frac{\partial \phi_{H_{\beta}}}{\partial \lambda_{i}} \tag{A.10}
\end{equation*}
$$

which is nothing but the minus sign of Eq.(27).

## Appendix B. A calculation of the inverse matrix of $K$

We calculate the inverse of the matrix $K$, each element of which was calculated in Appendix A. This calculation of $K^{-1}$ is also not restricted to affine systems. We assume that the matrix $b$ defined in Eq.(21) is regular. First let us set this inverse matrix in a blockwise form,

$$
K^{-1}=\left(\begin{array}{cccc}
V_{1,1} & V_{1,2} & V_{1,3} & V_{1,4}  \tag{B.1}\\
V_{2,1} & V_{2,2} & V_{2,3} & V_{2,4} \\
V_{3,1} & V_{3,2} & V_{3,3} & V_{3,4} \\
V_{4,1} & V_{4,2} & V_{4,3} & V_{4,4}
\end{array}\right)
$$

The first to the fourth columns of the product $K K^{-1}=$ $I_{2 n+2 m}(2 n+2 m$-dimensional unit matrix $)$ are written down as follows, respectively.

- The first column,

$$
\begin{align*}
\left(\begin{array}{cccc}
0 & 0 & I_{n} & -a \\
0 & 0 & 0 & -b \\
-I_{n} & 0 & 0 & -c \\
{ }^{t} a & { }^{t} b & { }^{t} c & 0
\end{array}\right)\left(\begin{array}{c}
V_{1,1} \\
V_{2,1} \\
V_{3,1} \\
V_{4,1}
\end{array}\right) & =\left(\begin{array}{c}
V_{3,1}-a V_{4,1} \\
-b V_{4,1} \\
-V_{1,1}-c V_{4,1} \\
t a V_{1,1}+{ }^{t} b V_{2,1}+{ }^{t} c V_{3,1}
\end{array}\right) \\
& =\left(\begin{array}{c}
I_{n} \\
0 \\
0 \\
0
\end{array}\right) . \tag{B.2}
\end{align*}
$$

From the second row we have $V_{4,1}=0$, which is substitued into the first and the third rows to obtain $V_{3,1}=I_{n}$ and $V_{1,1}=0$, respectively. These results are substituted into the fourth row to get $V_{2,1}=-\left({ }^{t} b\right)^{-1}{ }^{t} c$.

- The second column,

$$
\begin{align*}
\left(\begin{array}{cccc}
0 & 0 & I_{n} & -a \\
0 & 0 & 0 & -b \\
-I_{n} & 0 & 0 & -c \\
{ }^{t} a & { }^{t} b & { }^{t} c & 0
\end{array}\right)\left(\begin{array}{c}
V_{12} \\
V_{2,2} \\
V_{3,2} \\
V_{4,2}
\end{array}\right) & =\left(\begin{array}{c}
V_{3,2}-a V_{4,2} \\
-b V_{4,2} \\
-V_{1,2}-c V_{4,2} \\
t_{a V_{1,2}+{ }^{t} b V_{2,2}+{ }^{t} c V_{3,2}}
\end{array}\right) \\
& =\left(\begin{array}{c}
0 \\
I_{m} \\
0 \\
0
\end{array}\right) . \tag{B.3}
\end{align*}
$$

From the second row we have $V_{4,2}=-b^{-1}$, which is substitued into the first and the third rows to obtain $V_{3,2}=-a b^{-1}$ and $V_{1,2}=c b^{-1}$, respectively. These results are substituted into the fourth row to get $V_{2,2}=$ $\left({ }^{t} b\right)^{-1}\left({ }^{t} c a-{ }^{t} a c\right) b^{-1}$.

- The third column,

$$
\begin{align*}
\left(\begin{array}{cccc}
0 & 0 & I_{n} & -a \\
0 & 0 & 0 & -b \\
-I_{n} & 0 & 0 & -c \\
{ }^{t} a & { }^{t} b & { }^{t} c & 0
\end{array}\right)\left(\begin{array}{c}
V_{1,3} \\
V_{2,3} \\
V_{3,3} \\
V_{4,3}
\end{array}\right) & =\left(\begin{array}{c}
V_{3,3}-a V_{4,3} \\
-b V_{4,3} \\
-V_{1,3}-c V_{4,3} \\
t a V_{1,3}+{ }^{t} b V_{2,3}+{ }^{t} c V_{3,3}
\end{array}\right) \\
& =\left(\begin{array}{c}
0 \\
0 \\
I_{n} \\
0
\end{array}\right) . \tag{B.4}
\end{align*}
$$

From the second row we have $V_{4,3}=0$, which is substitued into the first and the third rows to obtain $V_{3,3}=0$ and $V_{1,3}=-I_{n}$, respectively. These results are substituted into the fourth row to get $V_{2,3}=\left({ }^{t} b\right)^{-1}{ }^{t} a$.

- The fourth column,

$$
\begin{align*}
\left(\begin{array}{cccc}
0 & 0 & I_{n} & -a \\
0 & 0 & 0 & -b \\
-I_{n} & 0 & 0 & -c \\
{ }^{t} a & { }^{t} b & { }^{t} c & 0
\end{array}\right)\left(\begin{array}{c}
V_{1,4} \\
V_{2,4} \\
V_{3,4} \\
V_{4,4}
\end{array}\right) & =\left(\begin{array}{c}
V_{3,4}-a V_{4,4} \\
-b V_{4,4} \\
-V_{1,4}-c V_{4,4} \\
t_{a V_{1,4}+{ }^{t} b V_{2,4}+{ }^{t} c V_{3,4}}
\end{array}\right) \\
& =\left(\begin{array}{c}
0 \\
0 \\
0 \\
I_{m}
\end{array}\right) . \tag{B.5}
\end{align*}
$$

From the second row we have $V_{4,4}=0$, which is substitued into the first and the third rows to obtain $V_{3,4}=0$ and $V_{1,4}=0$, respectively. These results are substituted into the fourth row to get $V_{2,4}=\left({ }^{t} b\right)^{-1}$.
These are summraized to obtain the inverse matrix as follows,

$$
K^{-1}=\left(\begin{array}{cc}
0 & c b^{-1} \\
-\left({ }^{t} b\right)^{-1}{ }^{t} c & \left({ }^{t} b\right)^{-1}\left({ }^{t} c a-{ }^{t} a c\right) b^{-1}  \tag{B.6}\\
I_{n} & \\
0 & -a b^{-1} \\
-b^{-1} \\
-I_{n} & 0 \\
\left({ }^{t} b\right)^{-1}{ }^{t} a & \left({ }^{t} b\right)^{-1} \\
0 & 0 \\
0 & 0
\end{array}\right) .
$$

## Appendix C. A derivation of the generalized Hamilton-Jacobi equation from the linear wave equation

We show that taking the real part projection of the linear wave equation imposed on the complex-valued wave function leads to the generalized Hamilton-Jacobi equation for the affine system.

We first note that the following formula holds, for arbitrary functions $f, g$ and $W$ of the state variable $\vec{x}$,

$$
\begin{align*}
& \overline{\nabla_{i}^{f} \nabla_{j}^{g}} W=\frac{\nabla_{i}^{f} \nabla_{j}^{g} W+\nabla_{j}^{g} \nabla_{i}^{f} W}{2} \\
&=f g \frac{\partial^{2} W}{\partial x_{i} \partial x_{j}} \\
&+\frac{1}{2}\left\{\left(f \frac{\partial g}{\partial x_{i}}+g \frac{\partial f}{\partial x_{i}}\right) \frac{\partial W}{\partial x_{j}}+\left(f \frac{\partial g}{\partial x_{j}}+g \frac{\partial f}{\partial x_{j}}\right) \frac{\partial W}{\partial x_{i}}\right\} \\
&\left.\quad+\frac{1}{4}\left(f \frac{\partial^{2} g}{\partial x_{i} \partial x_{j}}+\frac{\partial f}{\partial x_{i}} \frac{\partial g}{\partial x_{j}}+g \frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}\right) W . \text { (C. } 1\right) \tag{C.1}
\end{align*}
$$

So when the "kinetic energy" part of the Hamiltonian operator (56) is operated on the decomposed form of the wave function of Eq.(58), we have,

$$
\left(-\overline{\hat{u}_{\underline{\alpha}} R_{\underline{\alpha}, \underline{\beta}} \hat{u}_{\underline{\beta}}}-\overline{\hat{\lambda}_{\underline{i}} \hat{u}_{\underline{\alpha}} g_{\underline{i}, \underline{\underline{\alpha}}}}\right) \psi
$$

$$
\begin{gather*}
=-\frac{H_{R}^{2}}{2}\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}} \overline{\nabla_{\underline{i}}^{g_{i}, \underline{\alpha}} \nabla_{\underline{j}}^{g_{j}, \underline{\beta}}} \psi \\
=-\frac{H_{R}^{2}}{2}\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}}\left[g_{\underline{i}, \underline{\alpha}} g_{\underline{j}, \underline{\beta}} \frac{\partial^{2} \psi}{\partial x_{\underline{i}} \partial x_{\underline{j}}}\right. \\
+\frac{1}{2}\left\{\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{i}}}+g_{\underline{j}, \underline{\beta}} \frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{i}}}\right) \frac{\partial \psi}{\partial x_{\underline{j}}}\right. \\
\left.+\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{j}}}+g_{\underline{j}, \underline{\beta}} \frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{j}}}\right) \frac{\partial \psi}{\partial x_{\underline{i}}}\right\} \\
\left.+\frac{1}{4}\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial^{2} g_{j \underline{j}, \underline{\beta}}}{\partial x_{\underline{i}} \underline{x_{\underline{j}}}}+\frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{i}}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{j}}}+g_{\underline{j}, \underline{\beta}} \frac{\partial^{2} g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{i}} \partial x_{\underline{j}}}\right) \psi\right] . \tag{C.2}
\end{gather*}
$$

The operation of the "convex" term is calculated as,

$$
\begin{align*}
-\overline{\hat{\lambda}_{\underline{j}} F_{\underline{j}}(\vec{x})} \psi & =-i H_{R} \nabla_{\underline{j}}^{F_{\underline{j}}} \psi \\
& =-i H_{R}\left(F_{\underline{j}} \frac{\partial \psi}{\partial x_{\underline{j}}}+\frac{1}{2} \frac{\partial F_{\underline{j}}}{\partial x_{\underline{j}}} \psi\right) . \tag{C.3}
\end{align*}
$$

The partial derivatives of the wave function are expressed by the partial derivatives of the absolute value function and those of the phase function as follows,

$$
\begin{align*}
& \frac{\partial \psi}{\partial x_{i}}=\frac{\partial\left(R^{q} e^{\frac{i}{H_{R}} S^{q}}\right)}{\partial x_{i}} \\
& =\left(\frac{\partial R^{q}}{\partial x_{i}}+\frac{i}{H_{R}} R^{q} \frac{\partial S^{q}}{\partial x_{i}}\right) e^{\frac{i}{H_{R}} S^{q}},  \tag{C.4}\\
& \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{j}}=\left(\frac{\partial^{2} R^{q}}{\partial x_{i} \partial x_{j}}+\frac{i}{H_{R}} \frac{\partial S^{q}}{\partial x_{i}} \frac{\partial R^{q}}{\partial x_{j}}\right. \\
& +\frac{i}{H_{R}} \frac{\partial S^{q}}{\partial x_{j}} \frac{\partial R^{q}}{\partial x_{i}}+R^{q}\left(\frac{i}{H_{R}}\right)^{2} \frac{\partial S^{q}}{\partial x_{i}} \frac{\partial S^{q}}{\partial x_{j}} \\
& \left.+R^{q} \frac{i}{H_{R}} \frac{\partial^{2} S^{q}}{\partial x_{i} \partial x_{j}}\right) e^{\frac{i}{H_{R}} S^{q}},  \tag{C.5}\\
& \frac{\partial \psi}{\partial t}=\left(\frac{\partial R^{q}}{\partial t}+\frac{i}{H_{R}} R^{q} \frac{\partial S^{q}}{\partial t}\right) e^{\frac{i}{H_{R}} S^{q}} . \tag{C.6}
\end{align*}
$$

Substituting these Eqs.(C.4), (C.5) and (C.6) into Eq.(C.2), after multiplying $e^{-\frac{i}{H_{R}} S^{q}}$ and taking the real part, we have, for the kinetic part,

$$
\begin{gather*}
R e\left[\left(-{\overline{\hat{u}_{\underline{\alpha}}} R_{\underline{\alpha}, \underline{\beta}} \hat{u}_{\underline{\beta}}}-\overline{\hat{\lambda}}_{\underline{i}-\hat{u}_{\underline{\alpha}} g_{\underline{i}, \underline{\alpha}}}\right) \psi \cdot e^{-\frac{i}{H_{R}} S^{q}}\right] \\
=-\frac{H_{R}^{2}}{2}\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}} \\
{\left[g_{\underline{i}, \underline{,}} g_{\underline{j}, \underline{\beta}}\left(\frac{\partial^{2} R^{q}}{\partial x_{\underline{i}} \partial x_{\underline{j}}}+\left(\frac{i}{H_{R}}\right)^{2} R^{q} \frac{\partial S^{q}}{\partial x_{\underline{i}}} \frac{\partial S^{q}}{\partial x_{\underline{j}}}\right)\right.} \\
+\frac{1}{2}\left\{\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{i}}}+g_{\underline{j}, \underline{\beta}} \frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{i}}}\right) \frac{\partial R^{q}}{\partial x_{\underline{j}}}\right. \\
\left.+\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{j}}}+g_{\underline{j}, \underline{\beta}} \frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{j}}}\right) \frac{\partial R^{q}}{\partial x_{\underline{i}}}\right\} \\
\left.+\frac{1}{4}\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial^{2} g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{i}} \partial x_{\underline{j}}}+\frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{i}}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{j}}}+g_{\underline{j}, \underline{\beta}} \frac{\partial^{2} g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{i}} \partial x_{\underline{j}}}\right) R^{q}\right] \cdot(\mathrm{C} \tag{C.7}
\end{gather*}
$$

The convex and the time derivative terms are given as,

$$
\begin{equation*}
R e\left[-\overline{\hat{\lambda}_{\underline{j}} F_{\underline{j}}(\vec{x})} \psi \cdot e^{-\frac{i}{H_{R}} S^{q}}\right]=F_{\underline{j}} R^{q} \frac{\partial S^{q}}{\partial x_{\underline{j}}} \tag{C.8}
\end{equation*}
$$

$$
\begin{equation*}
R e\left[i H_{R} \frac{\partial \psi}{\partial t} e^{-\frac{i}{H_{R}} S^{q}}\right]=-R^{q} \frac{\partial S^{q}}{\partial t} \tag{C.9}
\end{equation*}
$$

respectively. Taking the real part of the linear wave equation (57) after multiplying $e^{-\frac{i}{H_{R}} S^{q}}$, we finally have,

$$
\begin{align*}
-R^{q} \frac{\partial S^{q}}{\partial t}= & \frac{1}{2}\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}} g_{\underline{i}, \underline{\alpha}} g_{\underline{j}, \underline{\beta}} R^{q} \frac{\partial S^{q}}{\partial x_{\underline{i}}} \frac{\partial S^{q}}{\partial x_{\underline{j}}} \\
& -V_{\operatorname{cost}} R^{q}+F_{\underline{j}} R^{q} \frac{\partial S^{q}}{\partial x_{\underline{j}}}-V^{q} R^{q} \tag{C.10}
\end{align*}
$$

where the additional cost $V^{q}$ is defined by Eq.(60). Dividing the above Eq.(C. 10) by $R^{q}$ leads to the generalized Hamilton-Jacobi equation (59).

In the following, we extract a partial differential equation for the absolute value function $R^{q}$ by taking the imaginary part of the wave equation. Substituting Eqs.(C.4) and (C.5) into Eq.(C. 2) and taking its imaginary part after multiplication of $e^{-\frac{i}{H_{R}} S^{q}}$, for kinetic energy part we have,

$$
\begin{align*}
& \operatorname{Im}\left[\left(-\hat{u}_{\underline{\alpha}} R_{\underline{\alpha}, \underline{\beta}} \hat{u}_{\underline{\beta}}\right.\right.\left.\left.\overline{\hat{\lambda}_{\underline{i}} \hat{u}_{\underline{\alpha}} g_{\underline{i}, \underline{\alpha}}}\right) \psi \cdot e^{-\frac{i}{H_{R}} S^{q}}\right] \\
&=- \frac{H_{R}^{2}}{2}\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}}\left[g _ { \underline { i } , \underline { \alpha } } g _ { \underline { j } , \underline { \beta } } \frac { 1 } { H _ { R } } \left(\frac{\partial R^{q}}{\partial x_{\underline{i}}} \frac{\partial S^{q}}{\partial x_{\underline{j}}}\right.\right. \\
&\left.+\frac{\partial S^{q}}{\partial x_{\underline{i}}} \frac{\partial R^{q}}{\partial x_{\underline{j}}}+R^{q} \frac{\partial^{2} S^{q}}{\partial x_{\underline{i}} \partial x_{\underline{j}}}\right) \\
&+\frac{1}{2 H_{R}}\left\{\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{i}}}+g_{\underline{j, \underline{\beta}}} \frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{i}}}\right) R^{q} \frac{\partial S^{q}}{\partial x_{\underline{j}}}\right. \\
&\left.\left.+\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{j}}}+g_{\underline{j}, \underline{\beta}} \frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{j}}}\right) R^{q} \frac{\partial S^{q}}{\partial x_{\underline{i}}}\right\}\right] . \quad \text { (C. } \tag{C.11}
\end{align*}
$$

The convex and the time derivative terms are given as,

$$
\begin{align*}
& \operatorname{Im}\left[-\overline{\hat{\lambda}_{\underline{j}} F_{\dot{j}}(\vec{x})} \psi \cdot e^{-\frac{i}{H_{R}} S^{q}}\right]= \\
& \quad-H_{R} F_{\underline{j}} \frac{\partial R^{q}}{\partial x_{\underline{j}}}-\frac{H_{R}}{2} \frac{\partial F_{\underline{j}}}{\partial x_{\underline{j}}} R^{q}  \tag{C.12}\\
& \operatorname{Im}\left[i H_{R} \frac{\partial \psi}{\partial t} e^{-\frac{i}{H_{R}} S^{q}}\right]=H_{R} \frac{\partial R^{q}}{\partial t} \tag{C.13}
\end{align*}
$$

respectively. Taking the imaginary part of the linear wave equation (57) after multiplying $e^{-\frac{i}{H_{R}} S^{q}}$ and dividing by the common factor $H_{R}$, we have,

$$
\begin{gather*}
\frac{\partial R^{q}}{\partial t}=-\frac{1}{2}\left(\bar{R}^{-1}\right)_{\underline{\alpha}, \underline{\beta}}\left[g _ { \underline { i } , \underline { \alpha } } g _ { \underline { j } , \underline { \beta } } \left(\frac{\partial R^{q}}{\partial x_{\underline{i}}} \frac{\partial S^{q}}{\partial x_{\underline{j}}}\right.\right. \\
+ \\
\left.+\frac{\partial S^{q}}{\partial x_{\underline{i}}} \frac{\partial R^{q}}{\partial x_{\underline{j}}}+R^{q} \frac{\partial^{2} S^{q}}{\partial x_{\underline{i}} \partial x_{\underline{j}}}\right) \\
+\frac{1}{2}\left\{\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{i}}}+g_{\underline{j}, \underline{\beta}} \frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{i}}}\right) R^{q} \frac{\partial S^{q}}{\partial x_{\underline{j}}}\right. \\
\left.\left.+\left(g_{\underline{i}, \underline{\alpha}} \frac{\partial g_{\underline{j}, \underline{\beta}}}{\partial x_{\underline{j}}}+g_{\underline{j}, \underline{,}} \frac{\partial g_{\underline{i}, \underline{\alpha}}}{\partial x_{\underline{j}}}\right) R^{q} \frac{\partial S^{q}}{\partial x_{\underline{i}}}\right\}\right]  \tag{C.14}\\
-F_{\underline{j}} \frac{\partial R^{q}}{\partial x_{\underline{j}}}-\frac{1}{2} \frac{\partial F_{\underline{j}}}{\partial x_{\underline{j}}} R^{q} .
\end{gather*}
$$

We note that in the above equation we have no $H_{R}$ both in r.h.s. and l.h.s..

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