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Kinematics of Manipulation Using the Theory of Polyhedral Convex Cones and its Application to Grasping and Assembly Operations

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A new approach to the kinematic analysis of object motion constrained by mechanical contacts is presented. In robotic manipulation, such as grasping and assembly, robots manipulate objects through mechanical contacts with the grasped object and with the environment. We need to understand the kinematic behavior of the object motion under the constraints by the mechanical contacts in order to find appropriate strategies for manipulation tasks.

In this paper, we first show that the constraints by mechanical contacts are generally described by a set of homogeneous linear inequalities. In task planning, it is often necessary for the planner to treat the complex inequalities. Thus, we develop an efficient mathematical tool based on the theory of polyhedral convex cones in order to treat the inequalities in a simple and systematic manner. Furthermore, we develop computation algorithms of the polyhedral convex cones in order to treat the inequalities on a computer. We apply the method to the planning of form-closure grasps, workpiece fixturing, and hybrid position/force control. Several examples demonstrate the usefulness of the algorithms.

Key Words: robot manipulator, manipulation, kinematics, task planning, polyhedral convex cones

1. Introduction

Kinematics and statics of arm linkages have been studied extensively in past decades. An arm linkage is a holonomic system consisting of multiple bodies, for which standard analytic methods have been established. In contrast, process models of manipulative tasks such as assembly are generally non-holonomic. Objects are in contact with each other and thereby constrained mechanically, but constraints are unidirectional since the objects may separate in one direction. The difference between the bidirectional and the unidirectional constraints is critical, since the latter refers to non-holonomic constraints, to which standard techniques do not apply.

In screw theory, unidirectional constraints have been characterized by repelling and reciprocal screws [1]. In grasp analysis, the unidirectional nature of constraints by fingers has been addressed in the analysis of form closure [2] and force closure [1),3]. Contacts between fingers and objects are also analyzed extensively [4)~6]. In fixture analysis, the accessibility and detachability conditions have been derived from a non-holonomic model of workpiece positioning [7]. Assembly tasks such as peg-into-hole mating are non-holonomic processes with unidirectional constraints. These assembly processes have been analyzed based on unidirectional constraint models [8),9].

In these papers, the unidirectional constraints are described by a set of inequalities or in some equivalent formulae. In these analyses, the intractable nature of inequalities creates difficulties; simultaneous inequalities are much harder to solve explicitly than equalities. Solutions are complex to represent and difficult to interpret. Unlike the solutions to simultaneous equations, the solutions to simultaneous inequalities are not given in an explicit, comprehensive, and understandable form, even if the inequalities are linear. This is a bottleneck in the analysis of manipulative tasks where objects are subject to unidirectional constraints.

The goal of this paper is to establish an underpinning mathematical tool for dealing with a variety of manipulative tasks that are governed by unidirectional constraints. We will introduce a coherent representation for formulating various problems including grasping, fixturing, and hybrid position/force control, all of which are treated as problems of solving a certain class of simultaneous inequalities. To solve these problems, we will develop a systematic method based on the theory of polyhedral convex cones. The new method not only produces simple, systematic solutions, but also provides a general perspective over many different problems and useful insight of non-holonomic systems with unidirectional constraints.

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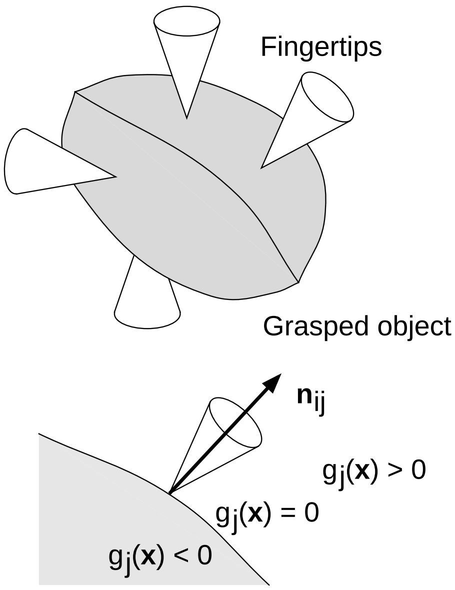

Fig. 1 Model of grasping by robotic hand

2. Description of Constrained Object Motion by Inequalities

2.1 Object Grasping by Robotic Hand

Grasp is to constrain an object by means of fingers that provide unidirectional constraints. Let us investigate the condition for an object to be totally constrained, regardless of friction between the fingertips and the object. This condition requires the fingers to surround an object so that no geometrically admissible displacement is allowed for the object. We deal with a rigid body consisting of a finite number of smooth surfaces, called face j. Let $g_j(\boldsymbol{x})$ be the distance between face j and an arbitrary point in space whose coordinates are $\boldsymbol{x}$ as illustrated in Figure 1. The distance function $g_j(\boldsymbol{x})$ is defined to be a signed distance so that it is negative inside the rigid body. We assume that all of the contacts are formulated by a finite number of point contacts.

Let $\boldsymbol{x}_i$ be the coordinates of the i-th vertex of the moving object. When the i-th vertex of the moving object is on the j-th face of the fixed object, the following equation is satisfied:

$$g_j(\boldsymbol{x}_i) = 0 \tag{1}$$

Let $\Delta\boldsymbol{x}_0$ is an infinitesimal displacement of the object position and $\Delta\boldsymbol{\theta}_0$ is that of the object orientation. When the moving object changes its location slightly, the signed distance between the i-th vertex and j-th face changes to:

$$d = g_j(\boldsymbol{x}_i - \Delta\boldsymbol{x}_0 - \Delta\boldsymbol{\theta}_0 \times \boldsymbol{x}_i), \tag{2}$$

where $\times$ represents the outer product of vectors. We assume that the function g_j is differentiable. Let $\boldsymbol{n}_{ij}$ be the inward normal vector of the j-th face at coordinates $\boldsymbol{x}_i$. Expanding eq.(2) and substituting eq.(1) into the expanded function, we have

$$\begin{aligned} d &= -\frac{\partial g_j}{\partial \boldsymbol{x}}(\boldsymbol{x}_i)(\Delta\boldsymbol{x}_0 + \Delta\boldsymbol{\theta}_0 \times \boldsymbol{x}_i) \\ &= -(\boldsymbol{n}_{ij})^T \Delta\boldsymbol{x}_0 - (\boldsymbol{x}_i \times \boldsymbol{n}_{ij})^T \Delta\boldsymbol{\theta}_0 \\ &= -(\boldsymbol{d}_{ij})^T \Delta\boldsymbol{q} \end{aligned} \tag{3}$$

where

$$\boldsymbol{d}_{ij} = \begin{bmatrix} \boldsymbol{n}_{ij} \\ \boldsymbol{x}_i \times \boldsymbol{n}_{ij} \end{bmatrix}, \tag{4}$$

$$\Delta\boldsymbol{q} = \begin{bmatrix} \Delta\boldsymbol{x}_0 \\ \Delta\boldsymbol{\theta}_0 \end{bmatrix}. \tag{5}$$

Since vertex i of the moving object lies on or outside the face j of the fixed object, the value of eq.(3) must be positive or equal to zero. Thus, the following condition must be satisfied:

$$\boldsymbol{d}_{ij}^T \Delta\boldsymbol{q} \le 0. \tag{6}$$

Contact conditions for other types of contact pairs can be expressed by an appropriate combination of inequalities in the form of eq.(6). For example, when a fingertip contacts with a concave edge, the condition is given by the intersection of two inequalities corresponding to adjacent faces of the concave edge. When a fingertip contacts with a convex edge, the condition is described by the union of two inequalities corresponding to adjacent faces of the convex edge.

The possible infinitesimal displacements of the object position and orientation must satisfy all of the conditions due to individual contact points. Expanding these conditions, a set of geometrically admissible displacements can be derived as:

$$A = \bigcup_{n=1}^{N} A_n \tag{7}$$

where

$$A_n = \{\Delta\boldsymbol{q} \mid (\boldsymbol{h}_{nm})^T \Delta\boldsymbol{q} \le 0,\ \forall m \in [1, M_n]\}, \quad \boldsymbol{h}_{nm} \in \{\boldsymbol{d}_{ij}\},\ \forall n, m. \tag{8}$$

Set A is referred to as *Admissible Displacement Set* in this paper. If the set A involves a non-zero element, the object is not geometrically constrained in that particular direction. Thus, the admissible displacement set A must be a set that has no elements other than $\mathbf{0}$ for the grasping. constraint is referred to as *Form Closure Grasp* [2].

2.2 Accessibility and Detachability of Workpiece

In assembly, a workpiece is positioned at a designated location relative to a fixed object. Fundamental questions

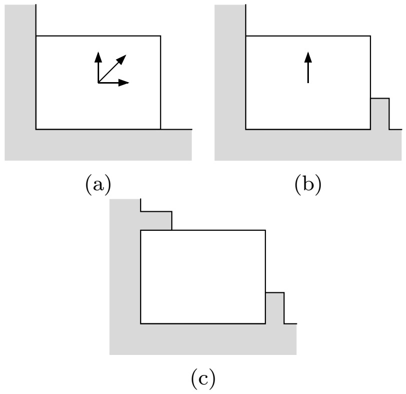

Fig. 2 Examples of planar object disassembly

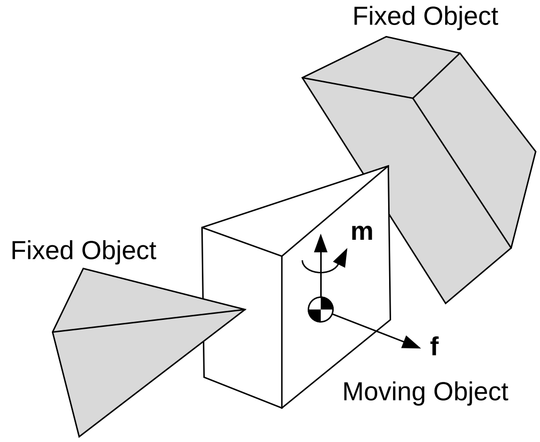

Fig. 3 Moving object constrained by contact with fixed objects

are to investigate whether the desired location is accessible for the workpiece and whether the workpiece is detachable from the fixture. Asada and By have formulated accessibility and detachability conditions by considering the local behavior of a workpiece in the vicinity of the designated location [7]. Workpiece in Figure 2-(a) and (b) is accessible and detachable in the vicinity of final location while workpiece in Figure 2-(c) is not accessible nor detachable.

Assuming that all the contacts are described by a finite number of point contacts, the conditions can be restated with regard to the admissible displacement set A. Namely, the workpiece is accessible and detachable in the vicinity of the designated location, if and only if a non-zero displacement $\Delta \boldsymbol{q}$ is involved in the set A.

$$\exists \Delta \boldsymbol{q} \neq 0 \ \text{ s.t. } \ \Delta \boldsymbol{q} \in A \tag{9}$$

Admissible translational displacements $\Delta \boldsymbol{x}$ are shown in the figure.

2.3 Hybrid Position/Force Control

In order to perform a task by using hybrid position/force control, we need to find the position-controlled space and the force-controlled space so that the robot motion may conform to the geometric constraints of the environment [10]. The position-controlled space in the hybrid control is equivalent to the space of admissible infinitesimal displacements, while the force-controlled space is the space of forces that satisfy the static equilibrium condition.

One moving object is in contact with fixed objects. The moving object is stable in the initial state. Assume that force $\boldsymbol{f}$ and moment $\boldsymbol{m}$ are applied to the moving object and that an infinitesimal displacement $\Delta \boldsymbol{q}$ occurs, as illustrated in Figure 3. Since we have no energy source except a robotic hand, work done by the hand $\Delta Work$ is equal to the sum of the increment of kinetic energy of the object, ΔT, and the increment of dispersing energy, $\Delta C \geq 0$. Since the moving object moves, the kinetic energy increases, say, $\Delta T > 0$. Thus, we have

$$\Delta Work = \boldsymbol{f} \cdot \Delta \boldsymbol{x} + \boldsymbol{m} \cdot \Delta \boldsymbol{\theta} = \boldsymbol{p}^T \boldsymbol{q} > 0 \tag{10}$$

where

$$\boldsymbol{p} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{m} \end{bmatrix}. \tag{11}$$

From the contraposition of the above discussion, we find that the moving object is stable if the following condition is satisfied:

$$\Delta Work = \boldsymbol{p}^T \boldsymbol{q} \leq 0 \tag{12}$$

Thus, the space of forces that satisfy the static equilibrium condition is described as follows:

$$F = \{\boldsymbol{p} \mid \boldsymbol{p}^T \Delta \boldsymbol{q} \leq 0, \ \forall \Delta \boldsymbol{q} \in A\}. \tag{13}$$

Namely, the force-controlled space is given by the above equation.

Thus, the above problems concerning assembly, grasps, and hybrid position/force control are all described with regard to a simultaneous system of linear inequalities. For these problems , we will develop a systematic computation method based on the theory of polyhedral convex cones.

3. Theory of Polyhedral Convex Cones

3.1 Definition of Polyhedral Convex Cones

All the problems discussed in the previous section are represented generally in the same form, that is, simultaneous inequalities in terms of inner products of two vectors. Problems associated with differential motions, or instantaneous kinematics and statics, are thus reduced to the problems of solving a simultaneous system of linear inequalities. From this section, we will develop a systematic method for solving these problems by applying the theory of polyhedral convex cones attributed to Goldman and Tucker [11].

Let $\boldsymbol{u}_1$ through $\boldsymbol{u}_k$ be k real vectors. A set of a linear combination of the vectors $\boldsymbol{u}_1$ through $\boldsymbol{u}_k$ with non-negative coefficients

$$A = \{\sum_{j=1}^{k} c_j \boldsymbol{u}_j \mid c_j \geq 0, \ \forall j \in [1, k]\}. \tag{14}$$

is called a *polyhedral convex cone* and is abbreviated to PCC. Note that the coefficients c_j are all non-negative and that in general the vectors $\boldsymbol{u}_j$ are not linearly independent. Since vectors $\boldsymbol{u}_1$ through $\boldsymbol{u}_k$ span the cone, we write the above equation simply by

$$A = span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}. \tag{15}$$

This form is referred to as the *Span Form* of the polyhedral convex cone, and each vector involved is called a span vector.

3.2 Solving Simultaneous Linear Inequalities

Let $\boldsymbol{a}_1$ through $\boldsymbol{a}_m$ be m real vectors. Let us consider a set of real vectors $\boldsymbol{x}$ given by

$$A = \{\boldsymbol{x} \mid \boldsymbol{a}_i^T \boldsymbol{x} \leq 0, \ \forall i \in [1, m]\}. \tag{16}$$

Set A represents a semi-infinite region surrounded by hyperplanes. Note that a vector $\boldsymbol{a}_i$ represents the normal to the i-th hyperplane. It has been proven that the above region is a polyhedral convex cone [11]. Namely, set A can be described as follows:

$$A = span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}.$$

Vectors $\boldsymbol{u}_1$ through $\boldsymbol{u}_k$ can be computed from vectors $\boldsymbol{a}_1$ through $\boldsymbol{a}_m$. For the sake of simplicity, the set given by eq.(16) is expressed as

$$A = face\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m\} \tag{17}$$

which is referred to as the *Face Form* of the polyhedral convex cone. Each vector involved is called a face vector.

3.3 Polar of Polyhedral Convex Cone

Let X be an arbitrary set of real vectors $\boldsymbol{x}$. The set defined by

$$X^* = \{\boldsymbol{y} \mid \boldsymbol{x}^T \boldsymbol{y} \leq 0, \ \forall \boldsymbol{x} \in X\} \tag{18}$$

is called the polar of the set X.

Let us consider the relationship between a polyhedral convex cone and its polar. Figure 4 illustrates a simple example of a two-dimensional polyhedral convex cone and its polar. The face form of the polyhedral convex cone A is given by $A = face\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. The cone A can be described in the span form as $A = span\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$. From the figure, we can find that the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ span the polar A^*. Namely, $A^* = span\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. Thus, we can describe the polar in the face form as $A^* = face\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

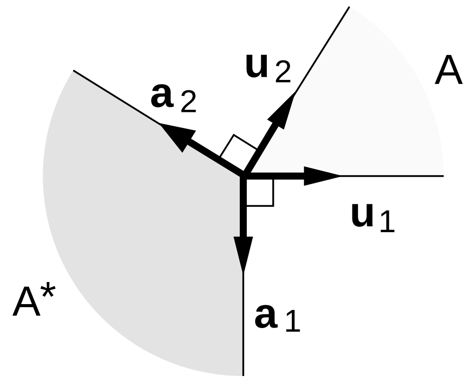

Fig. 4 Polyhedral convex cone and its polar

According to Goldman and Tucker, the following theorem is satisfied for an arbitrary polyhedral convex cone and its polar [11].

Theorem 1. The polar of a polyhedral convex cone $A = face\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m\}$ is given by a span form:

$$A^* = span\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m\}.$$

The polar of a polyhedral convex cone in span form $A = span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$ is given by a face form:

$$A^* = face\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}.$$

From this theorem, it follows that the polar of a polyhedral convex cone is a polyhedral convex cone as well. The polar A^* is referred to as the *dual polyhedral convex cone* of A. Note that the following property is derived from the above theorem.

$$(A^*)^* = A. \tag{19}$$

Let us consider the conversion between face and span forms. The problem is to derive a set of span vectors from a given set of face vectors, and vise versa. The conversion from face to span form can be performed by solving linear programming (LP) problems. The conversion from span to face form can also be performed by solving linear programming problems and using the above theorem. We first convert a polyhedral convex cone A to its dual polyhedral convex cone, and then solve linear programming problems in order to derive vectors $\boldsymbol{a}_1$ through $\boldsymbol{a}_m$ from vectors $\boldsymbol{u}_1$ through $\boldsymbol{u}_k$. Note that we regard $\boldsymbol{a}_i$ as a span vector and $\boldsymbol{u}_j$ as a face vector in the dual polyhedral convex cone. Thus, the conversion can be performed in both ways by simply solving linear programming problems. This algorithm is referred to as CONVERT.

3.4 Basic Properties of Polyhedral Convex Cones

Let X and Y be two sets of real vectors. The set defined by

$$X + Y = \{\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{x} \in X,\ \boldsymbol{y} \in Y\} \tag{20}$$

is called the convex sum of sets X and Y.

Polyhedral convex cones possess the following properties. The intersection of two polyhedral convex cones is also a polyhedral convex cone and is given by

$$\begin{aligned} & face\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m\} \cap face\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n\} \\ = \ & face\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m, \boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n\}. \end{aligned} \tag{21}$$

The convex sum of two polyhedral convex cones is also a polyhedral convex cone and is given by

$$\begin{aligned} & span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\} + span\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_l\} \\ = \ & span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k, \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_l\}. \end{aligned} \tag{22}$$

Applying the theorem to the above intersection and convex sum respectively, we can derive:

$$(A \cap B)^* = A^* + B^*, \tag{23}$$

$$(A + B)^* = A^* \cap B^*. \tag{24}$$

Consequently, the polar of a PCC, the intersection of PCC's, and the convex sum of PCC's are PCC's. The polar of dual PCC, the polar of the intersection, and the polar of the convex sum are given by eqs.(19), (23), and (24), respectively.

4. Methods for Solving the Inequality Problems

In this section, we will extend the theory of PCC's in order to obtain procedures for solving the inequality problems associated with assembly and grasps as described in Section 2. From the basic properties of PCC's, we can derive the following algorithms for the operations of PCC's.

DUAL(A) = compute the dual PCC of a given PCC.
Using Theorem 1, the dual PCC's can be obtained in both face and span forms.

INTERSECT(A,B) = compute the intersection of two PCC's, A and B.
If A and B are given in the span form, the algorithm CONVERT is first applied to the given PCC's in order to get face forms. For the face form PCC's, the intersection is directly obtained by eq.(21).

CONVEXSUM(A,B) = compute the convex sum of two PCC's, A and B.
If A and B are given in the face form, the algorithm CONVERT is first applied to the given PCC's to obtain span forms. For the span form PCC's, the convex sum is directly attained by eq.(22).

By using the above four algorithms, we can solve the fundamental inequality problems in a simple manner.

[1] The problem to examine whether a polyhedral convex cone A involves non-zero elements

If a polyhedral convex cone A is described in a face form, we apply algorithm CONVERT in order to describe it in the span form:

$$A = span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$$

An arbitrary non-zero elements $\boldsymbol{x}$ involved in A is described by a linear combination of span vectors. It implies that no non-zero elements are involved in A if $k = 0$. Thus, the polyhedral convex cone A involves non-zero elements, if and only if there exists span vectors of the polyhedral convex cone A:

$$A \neq \{\boldsymbol{0}\} \iff k \neq 0$$

The above method for examining whether a polyhedral convex cone involves non-zero elements is referred to as procedure NONZERO in this paper. Procedure NONZERO(A) returns a value of TRUE if a polyhedral convex cone A has non-zero elements and a value of FALSE otherwise.

[2] The problem to examine whether a vector $\boldsymbol{r}$ is involved in a polyhedral convex cone A

If a polyhedral convex cone A is described in a span form, we apply algorithm CONVERT in order to describe it in the face form:

$$A = face\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m\}$$

From the definition of face form, eq.(16), a vector $\boldsymbol{r}$ is involved in the polyhedral convex cone A if and only if

$$\boldsymbol{a}_i^T \boldsymbol{r} \leq 0, \quad \forall i \in [1, m]$$

is satisfied.

The above method for investigating whether a vector is involved in a polyhedral convex cone is referred to as procedure ELEMENT in this paper. Procedure ELEMENT($\boldsymbol{r}$,A) returns a value of TRUE if a vector $\boldsymbol{r}$ is involved in a polyhedral convex cone A and a value of FALSE otherwise.

[3] The problem to examine whether a polyhedral convex cone A is a subset of another polyhedral convex cone B

When a polyhedral convex cone A is described in a face form, we apply algorithm CONVERT in order to describe it in the span form:

$$A = span\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$$

Table 1 Algorithm to examine form closure grasp condition

```
for n := 1 to N do
begin
  A_n := face{h_n1, h_n2, ..., h_nM_n};
  if NONZERO(A_n) = TRUE then
    return(FALSE)
end;
return(TRUE)
```

Table 2 Procedure to compute admissible force set

```
A := {0};
for n := 1 to N
begin
  A_n := face{h_n1, h_n2, ..., h_nM_n};
  A := CONVEXSUM(A, A_n)
end;
F := DUAL(A)
```

The polyhedral convex cone A is a subset of another polyhedral convex cone B if and only if the following condition is satisfied:

$$\boldsymbol{u}_j \in B, \quad \forall j \in [1, m].$$

Using the procedure ELEMENT, we can find whether each vector $\boldsymbol{u}_j$ is involved in the polyhedral convex cone B. Thus, we can examine whether the above condition is satisfied or not. This method is referred to as procedure SUBSET in this paper. Procedure SUBSET(A,B) returns a value of TRUE if a polyhedral convex cone A is a subset of another cone B and a value of FALSE otherwise.

5. Applications to Planning of Manipulative Tasks

In this section, we apply the above methods to the manipulation problems described in Section 2.

5.1 Object Grasping

Using the notation introduced in Section 3, the admissible displacement set A of a grasped object is described by

$$A = \bigcup_{n=1}^{N} A_n \tag{25}$$

and

$$A_n = face\{\boldsymbol{h}_{n1}, \boldsymbol{h}_{n2}, \cdots, \boldsymbol{h}_{nM_n}\}. \tag{26}$$

The set A_1 through A_N are polyhedral convex cones. Thus, the admissible displacement set A is a union of polyhedral convex cones.

The condition for form closure grasps has been given by $A = \{\mathbf{0}\}$, which is equivalent to

$$A_n = \{\mathbf{0}\}, \quad \forall n \in [1, N]. \tag{27}$$

Using procedure NONZERO developed in the previous section, this condition is described as follows:

$$\text{NONZERO}(A_n) = \text{FALSE}, \quad \forall n \in [1, N] \tag{28}$$

The procedure to examine form closure grasps is listed in Table 1. This subroutine returns a value of TRUE if the form closure condition is met and a value of FALSE otherwise.

5.2 Accessibility and Detachability

The admissible displacement set A of a workpiece at a given final configuration is the same as eqs.(25) and (26). Thus, we can examine the accessibility/detachability condition using the procedure listed in Table 1.

5.3 Hybrid Position/Force Control

Let A be the admissible displacement set, which is the position-controlled space. Comparing eqs.(13) and (18), the admissible force set F is denoted as follows:

$$F = A^*. \tag{29}$$

In other words, the admissible force set F is the polar of the admissible displacement set A.

The admissible displacement set A is a union of polyhedral convex cones A_1 through A_N, as shown in eq.7:

$$A = A_1 \cup A_2 \cup \cdots \cup A_N$$

Note that the union of polyhedral convex cones is not always a polyhedral convex cone. The polar of the union is, however, a polyhedral convex cone. The following equation is satisfied for arbitrary polyhedral convex cones, X and Y:

$$(X \cup Y)^* = (X + Y)^*. \tag{30}$$

Applying eq.(30) into eq.(29), we have

$$\begin{aligned} F &= [A_1 \cup A_2 \cup \cdots \cup A_N]^* \\ &= [A_1 + A_2 + \cdots + A_N]^*. \end{aligned} \tag{31}$$

We find that the admissible force set F is the dual polyhedral convex cone of the convex sum of polyhedral convex cones A_1 through A_N. Using algorithm CONVEXSUM, we can compute the convex sum. Next, using algorithm DUAL, we can compute the polar F of the convex sum. Thus, we can compute the admissible force set F using the procedure shown in Table 2. The polyhedral convex cone F computed in this procedure gives the admissible force set.

It should be noted that both the admissible displacement set and the admissible force set are linear subspaces and orthogonal complements with each other when the geometric constraints are bidirectional [10]. On the other hand, both sets are not linear subspaces but a union of

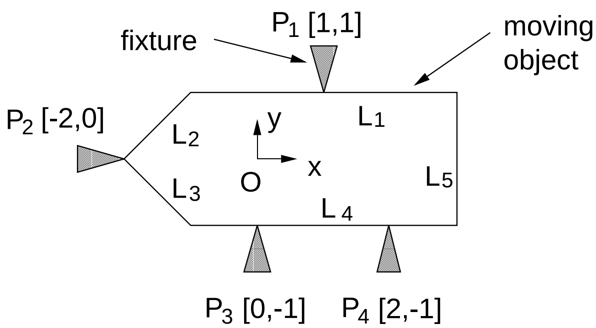

Fig. 5 Simple example of planar object and fixed points

PCC and its dual PCC when the constraints are unidirectional. Moreover, bidirectional constraints are holonomic while unidirectional constraints are non-holonomic.

We explain the key techniques of the computation by taking a simple example shown in Figure 5. The fixture is modeled by four points P_1 through P_4. Point P_1 is in contact with surface L_1, and points P_3 and P_4 with surface L_4. Point P_2 is in contact with a convex vertex defined as the intersection of surfaces L_2 and L_3. Let $\boldsymbol{x}_i$ be coordinates of point P_i and $\boldsymbol{n}_j$ be the outward normal vector of surface L_j. Let us compute the admissible displacement set A. Inequality conditions for displacement $\Delta\boldsymbol{q}$ to be admissible at individual contact points are derived as:

$$\begin{aligned}
&\boldsymbol{d}_{11}^T\Delta\boldsymbol{q} \le 0\\
&\boldsymbol{d}_{22}^T\Delta\boldsymbol{q} \le 0 \quad \text{or} \quad \boldsymbol{d}_{23}^T\Delta\boldsymbol{q} \le 0\\
&\boldsymbol{d}_{34}^T\Delta\boldsymbol{q} \le 0\\
&\boldsymbol{d}_{44}^T\Delta\boldsymbol{q} \le 0
\end{aligned}$$

where

$$\boldsymbol{d}_{ij} = \begin{bmatrix} \boldsymbol{n}_j \\ \boldsymbol{x}_i \times \boldsymbol{n}_j \end{bmatrix}. \tag{32}$$

Computing the value of vector $\boldsymbol{d}_{ij}$, we have

$$\begin{aligned}
\boldsymbol{d}_{11} &= [0,\ 1,\ 1]^T\\
\boldsymbol{d}_{22} &= [-1,\ 1,\ -2]^T\\
\boldsymbol{d}_{23} &= [-1,\ -1,\ 2]^T\\
\boldsymbol{d}_{34} &= [0,\ -1,\ 0]^T\\
\boldsymbol{d}_{44} &= [0,\ -1,\ -2]^T
\end{aligned}$$

Expanding the above inequalities, the admissible displacement set A is described by

$$A = A_1 \cup A_2 \tag{33}$$

where

$$\begin{aligned}
A_1 &= face\{\boldsymbol{d}_{11}, \boldsymbol{d}_{22}, \boldsymbol{d}_{34}, \boldsymbol{d}_{44}\},\\
A_2 &= face\{\boldsymbol{d}_{11}, \boldsymbol{d}_{23}, \boldsymbol{d}_{34}, \boldsymbol{d}_{44}\}.
\end{aligned}$$

Let us compute the admissible force set F from the admissible displacement set $A = A_1 \cup A_2$ by using the procedure listed in Table 2. The admissible force set F is then given by

$$F = face\{[1, 0, 0]^T\}. \tag{34}$$

Namely,

$$F = \{[f_x, f_y, m]^T \mid f_x \le 0\}, \tag{35}$$

where f_x and f_y are translational forces along the x- and y-axes, respectively, and m is a moment. This equation shows that while the translational force f_x is non-positive, the force acting upon the object by a robot is balanced with reaction forces against the fixed points P_1 through P_4 and the object is not accelerated. Describing the admissible force set in the span form, we have

$$F = span\{\boldsymbol{d}_{11}, \boldsymbol{d}_{34}, \boldsymbol{d}_{44}, [-1, 0, 0]^T\}. \tag{36}$$

Forces $\boldsymbol{d}_{11}$, $\boldsymbol{d}_{34}$, and $\boldsymbol{d}_{44}$ are balanced with reaction forces against the fixed points P_1, P_3, and P_4, respectively. Force $[-1, 0, 0]^T$ is balanced with a reaction force against point P_2. This example shows a case where a reaction force is generated between convex vertices though no reaction forces act usually at the contact point between convex vertices.

6. Concluding Remarks

In this paper, we have established an underpinning mathematical and computational method for dealing with manipulative tasks that are governed by unidirectional constraints. A variety of manipulation problems have been treated separately in different streams of robotics research. We have formulated those problems using a coherent mathematical tool, that is, the theory of polyhedral convex cones. We have extended the theory of polyhedral convex cones and have developed several procedures, which allow us to solve manipulation problems in a systematic and straightforward manner. In addition, use of a coherent mathematical representation allows us to obtain a general perspective over many different manipulation problems and to understand the fundamental nature of manipulative tasks, which are governed by unidirectional constraints.

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