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Relation between Spectrum Density and Wavelet Transform of Correlation Function

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This paper concerns properties of the wavelet transform of a correlation function from the viewpoint of spectrum analysis. First of all, a basic characteristic is derived as a relation between the wavelet transform and the corresponding spectrum density. The relation shows that the wavelet transform can be treated as an estimate of the spectrum density if a mother wavelet is chosen appropriately. Its bias from the true value is also obtained for this case. Moreover, a selection of the mother wavelet is considered to reduce the bias. This consideration provides validity of using the Gabor function as a mother wavelet for applications of the wavelet transform of the correlation function.

Key Words: wavelet transform, correlation function, spectrum estimation

1. Introduction

There are many applications of the continuous wavelet transform in various areas, for example, signal processing, health monitoring, measurement and system identification [1), 2)]. Through the applications, its usefulness is revealed widely. We focused the wavelet transform of a correlation function [3)] and have continued theoretical researches and application studies, for instance, dead time estimation of linear systems [4), 5)] and system identification of a boiler plant [6)].

Our preceding studies showed that the wavelet transform of the correlation fucntion can be treated as a kind of estimate of the corresponding spectrum density if a mother wavelet (analyzing wavelet) is a complex sinusoid ($e^{j\omega_p t}$) multiplied by a real window function, like the Gabor function (also known as Complex Morlet wavelet) [3)]. However, there was no study about essential properties of the estimate, such as a bias and a variance. It is important for both theory and applications to analyze these properties quantitatively.

In addition, it is expected that there is a strong relation between the wavelet transform of the correlation function and the spectrum density even when the mother wavelet is not a product of the sinusoid and the window function. The reason is the following: the wavelet transform is very similar to the windowed Fourier transform except for frequency dependence of window functions, and the Blackman-Tukey method, which is one of the spectrum estimation methods, is identical to the windowed Fourier transform of the correlation function. In order to prove the relation, we also have to investigate the case of using general mother wavelets.

By the way, there is another problem related mother wavelets. In our preceding studies, the Gabor function was employed as a mother wavelet to obtain the wavelet transform of the correlation function. Although it is empirically known that use of the Gabor function leads good results, there is no theoretical proof. If we analyze the relation between the wavelet transform of the correlation function and the spectrum density including the case of general mother wavelets, it should be possible to gives a theoretical validity of using the Gabor function.

This paper reveals the relation between the wavelet transform of the correlation function and the corresponding spectrum density. The derivation of the relation will provides the bias of the spectrum density estimated from the wavelet transform of the correlation function. In addition, we will prove that this bias is suppressed if a mother wavelet is a product of the complex sinusoid and specific window functions. That is, such a mother wavelet is appropriate when the continuous wavelet transform is applied to the spectrum analysis, in particular, if the clear relation is expected between the wavelet transform of the correlation function and the spectrum density.

This paper is organized as follows. In section 2, a basic relation is derived between the wavelet transform of the correlation function and the spectrum density. Then, we show the condition that the wavelet transform can be treated as an estimate of the spectrum density and describe the bias from the true value. Section 3 illustrates that the bias stated above is suppressed if and only if a mother wavelet is expressed as the complex sinusoid

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windowed by even functions. In section 4, a numerical experiment is carried out to confirm validity of our result. Section 5 concludes this paper and mentions further researches.

Notation: The set of real numbers will be represented by $\boldsymbol{R}$ and the set of complex numbers by $\boldsymbol{C}$. The asterisk * denotes complex conjugation.

The Fourier transform of a signal $x(t)$ is represented by $X(\omega)$ and its definition is the following.

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

An inner product over the space of square integrable functions $L^2(\boldsymbol{R})$ is defined by

$$\langle x, y \rangle = \int_{-\infty}^{+\infty} x^*(t) y(t) dt.$$

A cross correlation function between $x(t) \in \boldsymbol{R}$ and $y(t+t') \in \boldsymbol{R}$ is denoted by $\phi_{x(t),y(t+t')}(\tau)$. We employ the following definition in this paper.

$$\phi_{x(t),y(t+t')}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) y(t+t'+\tau) dt$$

Thus $\Phi_{x(t),y(t+t')}(\omega)$, which is the Fourier transform of $\phi_{x(t),y(t+t')}(\tau)$, is the cross spectrum density between $x(t)$ and $y(t+t')$. When $t' = 0$, they are simply denoted by $\phi_{xy}(\tau)$ and $\Phi_{xy}(\omega)$ respectively. Similarly, an auto correlation function and a power spectrum density of $x(t) \in \boldsymbol{R}$ are denoted by $\phi_{xx}(\tau)$ and $\Phi_{xx}(\omega)$ respectively.

A mother wavelet (or analyzing wavelet) is expressed by $\psi(t)$. In general, $\psi(t)$ is a complex function. Wavelet bases are represented by $\psi_{a,b}(t)$ and defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right).$$

The parameter a is the dilation parameter (or scaling parameter, scale parameter) and b is the location parameter (or translation parameter, shift parameter). The wavelet transform of $x(t)$ is defined by using these bases as follows.

$$\begin{aligned}\tilde{x}(a,b) &= \langle \psi_{a,b}(t), x(t) \rangle \\ &= \int_{-\infty}^{+\infty} x(t) \frac{1}{\sqrt{a}} \psi^*\left(\frac{t-b}{a}\right) dt\end{aligned}$$

In this paper, the dilation parameter a is limited to positive real numbers and the domain of the shift parameter b is $\boldsymbol{R}$.

2. Wavelet Transform of Correlation Function and Spectrum Density

First, we will show a basic relation between the wavelet transform of a cross correlation function and the corresponding cross spectrum. Next, a discussion will be given for the case that the wavelet transform can be regarded as an estimate of the spectrum.

2.1 Basic Relation between Wavelet Transform of Correlation Function and Spectrum Density

This subsection describes a basic analysis of a relation between the wavelet transform of a correlation function and the corresponding spectrum density.

Let $\tilde{\phi}_{xy}(a,b)$ denote the wavelet transform of $\phi_{xy}(\tau)$, which is the cross correlation function between $x(t)$ and $y(t)$. The definition of the wavelet transform directly yields

$$\tilde{\phi}_{xy}(a,b) = \langle \psi_{a,b}(\tau), \phi_{xy}(\tau) \rangle. \qquad (1)$$

The next theorem provides the relation between $\tilde{\phi}_{xy}(a,b)$ and the cross spectrum density.

Theorem 1. Assume that ω_0 is an arbitrary constant and a cross spectrum density $\Phi_{xy}(\omega)$ is n-times differentiable in $\boldsymbol{R}$. Then,

$$\begin{aligned}\tilde{\phi}_{xy}(a,b) &= \frac{\sqrt{2\pi}\psi^*(0)}{\sqrt{a}} \Phi_{x(t),y(t+b)}\left(\frac{\omega_0}{a}\right) \\ &\quad + \sum_{k=1}^{n-1} Q_k + R_n, \end{aligned} \qquad (2)$$

where

$$Q_k = \frac{\sqrt{a}}{k!} \Phi^{(k)}_{x(t),y(t+b)}\left(\frac{\omega_0}{a}\right) \int_{-\infty}^{+\infty} \lambda^k \Psi^*(\omega_0 + a\lambda) d\lambda. \qquad (3)$$

In addition, $|R_n|$ is bounded as follows.

$$\begin{aligned}|R_n| &\le \frac{\sqrt{a}}{n!} \max_{\omega} \left| \Phi^{(n)}_{x(t),y(t+b)}(\omega) \right| \\ &\quad \times \int_{-\infty}^{+\infty} |\lambda^n \Psi^*(\omega_0 + a\lambda)| d\lambda \end{aligned} \qquad (4)$$

Proof :

$$\begin{aligned}\tilde{\phi}_{xy}(a,b) &= \langle \psi_{a,b}(\tau), \phi_{xy}(\tau) \rangle \\ &= \langle \psi_{a,0}(\tau - b), \phi_{x(t),y(t+b)}(\tau - b) \rangle \\ &= \langle \psi_{a,0}(\tau), \phi_{x(t),y(t+b)}(\tau) \rangle\end{aligned}$$

Since inner products satisfy the following identity

$$\int_{-\infty}^{+\infty} x^*(t) y(t) dt = \int_{-\infty}^{+\infty} X^*(\omega) Y(\omega) d\omega,$$

we obtain

$$\tilde{\phi}_{xy}(a,b) = \sqrt{a} \int_{-\infty}^{+\infty} \Psi^*(a\omega) \Phi_{x(t),y(t+b)}(\omega)\, d\omega. \qquad (5)$$

This equation is transformed as follows by applying $\omega = \omega_0/a + \lambda$.

$$\begin{aligned}&\tilde{\phi}_{xy}(a,b) = \\ &\quad \sqrt{a} \int_{-\infty}^{+\infty} \Psi^*(\omega_0 + a\lambda) \Phi_{x(t),y(t+b)}\left(\frac{\omega_0}{a} + \lambda\right) d\lambda\end{aligned} \qquad (6)$$

The Taylor series of $\Phi_{x(t),y(t+b)}(\omega_0/a+\lambda)$ at ω_0/a is

$$\begin{aligned}\Phi_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}+\lambda\Big) &= \Phi_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}\Big)\\ &+\sum_{k=1}^{n-1}\frac{\lambda^k}{k!}\Phi^{(k)}_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}\Big)\\ &+\frac{1}{n!}\Phi^{(n)}_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}+\xi(\lambda)\Big), \qquad (7)\end{aligned}$$

where $0<\xi(\lambda)<\lambda$ for $\lambda>0$ and $\lambda<\xi(\lambda)<0$ for $\lambda<0$. By substituting this into the equation (6), we have

$$\begin{aligned}\tilde{\phi}_{xy}(a,b) &= \sqrt{a}\Phi_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}\Big)\int_{-\infty}^{+\infty}\Psi^*(\omega_0+a\lambda)d\lambda\\ &+\sqrt{a}\sum_{k=1}^{\infty}\frac{1}{k!}\Phi^{(k)}_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}\Big)\int_{-\infty}^{+\infty}\lambda^k\Psi^*(\omega_0+a\lambda)d\lambda\\ &+\frac{\sqrt{a}}{n!}\int_{-\infty}^{+\infty}\Phi^{(n)}_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}+\xi(\lambda)\Big)\lambda^n\Psi^*(\omega_0+a\lambda)d\lambda. \qquad (8)\end{aligned}$$

Let R_n denote the last term of this equation. Thus the equation (2) is obtained by using

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Psi^*(\omega_0+a\lambda)d\lambda=\frac{\psi^*(0)}{a}.$$

The upper bound of R_n can be also derived from

$$\begin{aligned}&\left|\int_{-\infty}^{+\infty}\Phi^{(n)}_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}+\xi(\lambda)\Big)\lambda^n\Psi^*(\omega_0+a\lambda)d\lambda\right|\\ &\le\int_{-\infty}^{+\infty}\left|\Phi^{(n)}_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}+\xi(\lambda)\Big)\right||\lambda^n\Psi^*(\omega_0+a\lambda)|d\lambda\\ &\le\max_{\omega}\left|\Phi^{(n)}_{x(t),y(t+b)}(\omega)\right|\int_{-\infty}^{+\infty}|\lambda^n\Psi^*(\omega_0+a\lambda)|d\lambda. \qquad (9)\end{aligned}$$

∎

When $\Phi_{xy}(\omega)$ is infinitely differentiable, the following result is obtained similarly.

Corollary 1. Assume that ω_0 is an arbitrary constant. If $\Phi_{xy}(\omega)$ is infinitely differentiable,

$$\tilde{\phi}_{xy}(a,b)=\frac{\sqrt{2\pi}\psi^*(0)}{\sqrt{a}}\Phi_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}\Big)+R, \qquad (10)$$

where

$$\begin{aligned}R &= \sum_{k=1}^{\infty}Q_k\\ &= \sum_{k=1}^{\infty}\frac{\sqrt{a}}{k!}\Phi^{(k)}_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}\Big)\int_{-\infty}^{+\infty}\lambda^k\Psi^*(\omega_0+a\lambda)d\lambda \qquad (11)\end{aligned}$$

□

2.2 Spectrum Density Estimation by Wavelet Transform of Correlation Function and Its Bias

Suppose that Q_k and R_n in the equation (2) are sufficiently small. For Corollary 1, suppose that R in the equation (10) is so. Then

$$\frac{\sqrt{a}}{\sqrt{2\pi}\psi^*(0)}\tilde{\phi}_{xy}(a,b)\approx\Phi_{x(t),y(t+b)}\Big(\frac{\omega_0}{a}\Big). \qquad (12)$$

This implies that the left hand of the equation can be regarded as an estimate of the cross spectrum density between $x(t)$ and $y(t+b)$. (1).

The bias of this estimate from the true spectrum density is

$$\frac{\sqrt{a}}{\sqrt{2\pi}\psi^*(0)}\left(\sum_{k=1}^{n-1}Q_k+R_n\right) \qquad (13)$$

for the equation (2) and

$$\frac{\sqrt{a}}{\sqrt{2\pi}\psi^*(0)}R=\frac{\sqrt{a}}{\sqrt{2\pi}\psi^*(0)}\sum_{k=1}^{\infty}Q_k \qquad (14)$$

for the equation (10). If the approximation of the equation (12) is proper, we can treat the wavelet transform of the cross correlation as the estimate of the corresponding spectrum density. It requires that the equation (13) or (14) is small enough.

In order to make (13) and (14) small, it is sufficient that the magnitude of Q_k, defined in the equation (3), is small for each k. Consider the terms which constitute Q_k. It is impossible to adjust the k-th derivative of $\Phi_{x(t),y(t+b)}(\omega)$ since it depends on the spectrum to be estimated. In contrast, the absolute value of

$$\int_{-\infty}^{+\infty}\lambda^k\Psi^*(\omega_0+a\lambda)d\lambda \qquad (15)$$

can be smaller by choosing a mother wavelet $\psi(t)$ and ω_0 adequately. As a result, this suppresses the magnitude of Q_k. Further discussions about these choices will be shown in the next section.

As stated above, we can make the bias smaller for an unknown object by reducing the magnitude of the equation (15). However, to suppress $|Q_k|$, it is also necessary that the magnitude of

$$\Phi^{(k)}_{x(t),y(t+b)}\left(\frac{\omega_0}{a}\right) \qquad (16)$$

is not too large. This brings another problem. The magnitude is dependent on the location parameter b. It can be seen from $\Phi_{x(t),y(t+b)}(\omega)=e^{j\omega b}\Phi_{xy}(\omega)$. The derivative and high order derivatives of $\Phi_{x(t),y(t+b)}(\omega)$ may be very large depending on the value of b. Hence we must be aware that not all b allow use of the approximation (12). The range of b which suppress the magnitude of (16) depends on the object to be estimated. Therefore we don't

(1) This result matches the one of the article 3), though the difference of the Fourier transform definition causes the difference of the constant multiplier $\sqrt{2\pi}$.

discuss it anymore and leave it for further studies. For the analyses in the rest of the paper, the location parameter b is assumed to be such that the equation (16) is not too large.

3. Analysis for Windowed Complex Sinusoid Wavelets

This section will focus mother wavelets which can be expressed as a product of a even function $w(t)$ and a complex sinusoid, *i.e.*

$$\psi(t) = w(t)e^{j\omega_p t}. \tag{17}$$

We shall call these wavelets the *windowed complex sinusoid wavelets* in this paper. The main purpose of this section is to advance the analysis stated in the previous section for the wavelets. When $w(t) = e^{-t^2}$, the wavelet is known as the Gabor wavelet (Complex Morlet wavelet). It is employed often in wavelet analysis applications and its usefulness is confirmed experimentally.

Section 2. 2 illustrated that the wavelet transform of a cross correlation function can be considered as an estimate of the corresponding cross spectrum density (Eq. (12)) if the bias of the equation (13) or (14) is small. To do this, a mother wavelet $\psi(t)$ must be selected appropriately. We will show that the windowed complex sinusoid wavelets are proper for this purpose. That is to say, if the wavelets are employed, the wavelet transform of the cross correlation is equivalent to the estimate of the cross spectrum. In addition, its inverse is also true. If a mother wavelet is chosen such that $|Q_k|$ is small in a sense and the bias is suppressed, it becomes a windowed complex sinusoid wavelet. This implies that it is desirable to select the windowed complex sinusoid as the mother wavelet if the connection is emphasized between the wavelet transform of the correlation function and the spectrum density.

First, we explain the windowed complex sinusoid wavelet and define its center frequency. Next, validity of the definition is shown from the viewpoint that it suppress the bias considered in the section 2. 2. The proof of the validity will also reveal that the wavelets reduce the bias. Next, we will show that if a mother wavelet is chosen such that the bias is suppressed, then it becomes a windowed complex sinusoid wavelet. Finally, a relation with the Blackman-Tukey method will be discussed.

3. 1 Windowed Complex Sinusoid Wavelet and Center Frequency

As stated before, the windowed complex sinusoid mother wavelets are represented by

$$\psi(t) = w(t)e^{j\omega_p t}, \tag{18}$$

where $w(t) \in \boldsymbol{R}$ is an even function. Let us call ω_p the center frequency of the mother wavelet.

If a function $w(t)$ is a real and even function, its Fourier transform $W(\omega)$ is also real and even. From the equation (18), the Fourier transform of $\psi(t)$ satisfies

$$\Psi(\omega) = W(\omega - \omega_p). \tag{19}$$

Therefore $\Psi(\omega)$ is symmetrical about $\omega = \omega_p$ and ω_p is the center frequency of the mother wavelet. Also note that $\Psi(\omega)$ is real function.

The Gabor function [3), 8)] (Complex Morlet wavelet) is one of the typical this type mother wavelets. If the Hamming or Hanning window, commonly used in the spectrum analysis [9)], are chosen as $w(t)$, the mother wavelet also becomes this type wavelet.

It is desirable to choose a positive real function as $\Psi(\omega)$ when the wavelet transform of a correlation function is treated as an estimate of a spectrum density. This can be understood from the equation (5), which shows that $\tilde{\phi}_{xy}(a, b)$ is a weighted moving average of the spectrum density $\Phi_{x(t),y(t+b)}(\omega)$ and $\Psi^*(\omega)$ is its weighting function. Thus negative $\Psi(\omega)$ may cause troubles, for example, an estimated power spectrum becomes negative and an estimated cross spectrum has strange phase values. For the above reason, it is better to use positive real $\Psi(\omega)$.

3. 2 Validity of Definition of Center Frequency

It is possible to justify the choice of the center frequency in the section 3. 1 from another viewpoint, that is, the choice also suppresses the bias in the section 2. 2. Consider the integration

$$\int_{-\infty}^{+\infty} \lambda^k \Psi^*(\omega_0 + a\lambda) d\lambda \tag{20}$$

included in the equation (3). It can be proven that for each k, ω_0 which minimizes the magnitude of the integration is coincident with the center frequency ω_p defined in the section 3. 1. Suppressing the integral (20) makes the considered bias small since the sum of Q_k are proportional to the equation (13) and (14), which are the bias when the wavelet transform of a correlation function is treated as an estimate of a spectrum density. We will show the result as the following theorem.

Theorem 2. For the mother wavelet of the equation (17), ω_0 such that minimize

$$I_k(\omega_0) = \left| \int_{-\infty}^{+\infty} \lambda^k |\Psi^*(\omega_0 + a\lambda)| d\lambda \right| \tag{21}$$

is the same as the center frequency ω_p in the section 3. 1.

Proof: From the equation (19),

$$I_k(\omega_0) = \left| \int_{-\infty}^{+\infty} \lambda^k |W^*(\omega_0 - \omega_p + a\lambda)| d\lambda \right|. \tag{22}$$

Now let $\omega = \omega_0 - \omega_p$ and define $I'_k(\omega) = I_k(\omega_0 - \omega_p)$. Then we obtain

$$I'_k(\omega) = \left| \int_{-\infty}^{+\infty} \lambda^k |W^*(\omega + a\lambda)| d\lambda \right|. \quad (23)$$

Apply the transformation $\lambda' = \omega + a\lambda$ to this equation and let us reuse λ to represent λ' after the translation. Since $a > 0$ is assumed, thus

$$I'_k(\omega) = \left| \int_{-\infty}^{+\infty} \left(\frac{\lambda - \omega}{a}\right)^k |W^*(\lambda)| \frac{d\lambda}{a} \right|$$
$$= \frac{1}{a^{k+1}} \left| \int_{-\infty}^{+\infty} (\lambda - \omega)^k |W^*(\lambda)| d\lambda \right|. \quad (24)$$

i) k is odd: It is trivial that $I'_k(\omega)$ is zero and minimum at $\omega = 0$, that is, when $\omega_0 = \omega_p$. This is easily derived by the fact that the integrand is odd function.

ii) k is even: Let us calculate $a^{k+1} I'_k(\omega)$ first. When k is even, the integrand is always positive. It follows that

$$a^{k+1} I'_k(\omega) = \left| \int_{-\infty}^{0} (\lambda - \omega)^k |W^*(\lambda)| d\lambda \right| + \left| \int_{0}^{+\infty} (\lambda - \omega)^k |W^*(\lambda)| d\lambda \right|. \quad (25)$$

Apply the transformation $\lambda' = -\lambda$ to the first integration and recall that k is even and $W(\omega)$ is even function. Then

$$a^{k+1} I'_k(\omega) = \left| \int_{+\infty}^{0} (-\lambda' - \omega)^k |W^*(-\lambda')| d(-\lambda') \right| + \left| \int_{0}^{+\infty} (\lambda - \omega)^k |W^*(\lambda)| d\lambda \right| \quad (26)$$
$$= \left| \int_{0}^{+\infty} (\lambda' + \omega)^k |W^*(\lambda')| d\lambda' \right| + \left| \int_{0}^{+\infty} (\lambda - \omega)^k |W^*(\lambda)| d\lambda \right| \quad (27)$$
$$= \left| \int_{0}^{+\infty} ((\lambda + \omega)^k + (\lambda - \omega)^k) |W^*(\lambda)| d\lambda \right|. \quad (28)$$

We can calculate $a^{k+1} I'_k(0)$ similarly.

$$a^{k+1} I_k(0) = \left| \int_{0}^{+\infty} 2\lambda^k |W^*(\lambda)| d\lambda \right| \quad (29)$$

Since k is even,

$$(\lambda + \omega)^k + (\lambda - \omega)^k - 2\lambda^k = 2 \sum_{l=1}^{k/2} {}_kC_{2l} \lambda^{2l} \omega^{k-2l} \geq 0. \quad (30)$$

Integrands in both the equations (28) and (29) are always positive. Therefore

$$I'_k(\omega) - I'_k(0) = \frac{1}{a^{k+1}} \int_{0}^{+\infty} 2 \sum_{l=1}^{k/2} {}_kC_{2l} \lambda^{2l} \omega^{k-2l} |W^*(\lambda)| d\lambda \geq 0. \quad (31)$$

The equality holds if $\omega = 0$, *i.e.* $\omega_0 = \omega_p$. Hence $\omega_0 = \omega_p$ minimizes $I_k(\omega_0)$. ∎

We chose ω_p in the equation (17) for the center frequency of the mother wavelets. As seen above, this reduces Q_k and R_n of the equation (2) and R in the equation (10) and, in consequence, the bias is suppressed when the wavelet transform of the correlation function is treated as an estimate of the spectrum density. Also in this sense, the choice has validity.

The integration (20) coincides with (21) if $\Psi(\omega)$ is positive. Hence, ω_0 such that minimize

$$\left| \int_{-\infty}^{+\infty} \lambda^k \Psi^*(\omega_0 + a\lambda) d\lambda \right| \quad (32)$$

is the center frequency ω_p defined in the section 3. 1.

3. 3 Windowed Complex Sinusoid Wavelet and Bias

As we saw in the proof of Theorem 2, the equation (20) becomes zero if k is odd. This implies that Q_k of the equation (2) disappear for odd k. The windowed complex sinusoid wavelets suppress the bias in this sense, too.

Moreover, its inverse also holds. Suppose that a mother wavelet is chosen so that Q_k is zero for odd k, *i.e.*

$$\int_{-\infty}^{+\infty} \lambda^{2l+1} \Psi^*(\omega_p + a\lambda) d\lambda = 0, \quad l = 0, 1, \cdots. \quad (33)$$

Then, the wavelet becomes a windowed complex sinusoid.

Theorem 3. The equation (33) is satisfied and $\Psi(\omega)$ is real if and only if $\psi(t)$ is the windowed complex sinusoid mother wavelet.

Proof:

(Sufficiency) It is already shown in the section 3. 1 that $\Psi(\omega)$ is real if $\psi(t)$ is the windowed complex sinusoid mother wavelet. The proof of Theorem 2 also shows that the equation (33) holds under the same condition.

(Necessity) Let us define $\Psi_e(\omega)$ and $\Psi_o(\omega)$ as

$$\Psi_e(\omega) = \frac{1}{2} (\Psi(\omega) + \Psi(-\omega + 2\omega_p)) \quad (34)$$
$$\Psi_o(\omega) = \frac{1}{2} (\Psi(\omega) - \Psi(-\omega + 2\omega_p)). \quad (35)$$

These are defined to satisfy the following equations.

$$\Psi(\omega) = \Psi_e(\omega) + \Psi_o(\omega) \quad (36)$$
$$\Psi_e(\omega_p + \omega) = \Psi_e(\omega_p - \omega) \quad (37)$$
$$\Psi_o(\omega_p + \omega) = -\Psi_o(\omega_p - \omega) \quad (38)$$

It means that $\Psi(\omega)$ is decomposed into two components $\Psi_e(\omega)$ and $\Psi_o(\omega)$, and the formar is symmetric about $\omega = \omega_p$ and the latter is anti-symmetric.

Since the above decomposition gives

$$\int_{-\infty}^{+\infty} \lambda^{2l+1} \Psi_e^*(\omega_p + a\lambda) d\lambda = 0$$

$$\int_{-\infty}^{+\infty} \lambda^{2l} \Psi_o^*(\omega_p + a\lambda) d\lambda = 0,$$

the following equations are satisfied.

$$\int_{-\infty}^{+\infty} \lambda^{2l+1} \Psi^*(\omega_p + a\lambda) d\lambda = \int_{-\infty}^{+\infty} \lambda^{2l+1} \Psi_o^*(\omega_p + a\lambda) d\lambda \quad (39)$$

$$\int_{-\infty}^{+\infty} \lambda^{ln} \Psi^*(\omega_p + a\lambda) d\lambda = \int_{-\infty}^{+\infty} \lambda^{2l} \Psi_e^*(\omega_p + a\lambda) d\lambda \quad (40)$$

The equations imply that the equation (33) holds if $\Psi_o(\omega)$ is set to zero. Thus, it is sufficient to let $\Psi(\omega) = \Psi_e(\omega)$ and choose $\psi(t)$ satisfying

$$\Psi(\omega_p + \omega) = \Psi(\omega_p - \omega). \quad (41)$$

Next, consider $\psi(t)$ which satisfies this equation. From the inverse Fourier transform,

$$\begin{aligned} \psi(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(\omega) e^{j\omega t} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(\omega + \omega_p) e^{j(\omega + \omega_p)t} d\omega \\ &= \frac{1}{\sqrt{2\pi}} e^{j\omega_p t} \int_{-\infty}^{+\infty} \Psi(\omega + \omega_p) e^{j\omega t} d\omega. \end{aligned} \quad (42)$$

Now, let us suppose that

$$w(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(\omega + \omega_p) e^{j\omega t} d\omega. \quad (43)$$

Then, $\psi(t) = w(t)e^{j\omega_p t}$. We will show that $w(t)$ becomes even function, that is, $w(-t)$ coincides with $w(t)$. From the previous equation,

$$w(-t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(\omega + \omega_p) e^{j\omega(-t)} d\omega.$$

By using the transformation $\omega' = -\omega$ and the equation (41) , we have

$$\begin{aligned} w(-t) &= \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} \Psi(-\omega' + \omega_p) e^{j\omega' t} d\omega' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(-\omega' + \omega_p) e^{j\omega' t} d\omega' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(\omega' + \omega_p) e^{j\omega' t} d\omega' \\ &= w(t). \end{aligned}$$

Hence $w(t)$ is even function.

On the other hand, if $\Psi(\omega)$ is a real function, that is when $W(\omega)$ is real (recall $W(\omega) = \Psi(\omega + \omega_p)$), then $w(t) = w^*(-t)$. From this and the fact that $w(t)$ is even, $w(t)$ is a real function. ∎

As seen in this proof, Q_k depends on only $W_o(\omega)$ for odd k (the equation (39)), and only $W_e(\omega)$ for even k (the equation (40)). This fact yields the next corollary.

Corollary 2. Assume that l is natural number and $W_e(\omega) \neq 0$. For a given window function $w(t)$ such that its Fourier transform isn't even function, there exists another window function $w'(t)$ such that the same Q_{2l}, but $Q_{2l+1} = 0$, and its Fourier transform satisfies $W'(\omega) = W_e(\omega)$. □

The corollary states that we can select a mother wavelet and the related window function such that $Q_k = 0$ for odd k, without any other effects from the viewpoint of the bias. In other words, letting $Q_k = 0$ for odd k doesn't alter the value of Q_k for even k.

Consequently, the windowed complex sinusoid is a desirable mother wavelet to relate the wavelet transform of the cross correlation function with the cross spectrum density.

3.4 Relation with Blackman-Tukey Method

This subsection will treat the relation between two estimates of a cross spectrum density: one is calculated from a wavelet transform of a cross correlation function and another is by the Blackman-Tukey method. In particular, we will discuss their biases mainly.

The window function treated in this section, $w(t)$, is quite similar to the one in the Blackman-Tukey method if the condition $w(0) = 1$ is added. The only difference is that $w(t)$ used in the Blackman-Tukey method must satisfy one more condition: there exists a $M > 0$ such that $w(\tau) = 0$ for τ satisfying $|\tau| > M$. Since $w(0) = 1$ means $\psi^*(0) = 1$, the equation (12) becomes simpler as follows.

$$\frac{\sqrt{a}}{\sqrt{2\pi}} \tilde{\phi}_{xy}(a, b) \approx \Phi_{x(t),y(t+b)}\left(\frac{\omega_p}{a}\right) \quad (44)$$

Suppose that a mother wavelet is the windowed complex sinusoid wavelet and it also satisfies $w(0) = 1$. Then,

$$Q_2 = -\frac{\sqrt{2\pi}}{2\sqrt{a}} \frac{w''(0)}{a^2} \Phi''_{x(t),y(t+b)}\left(\frac{\omega_p}{a}\right). \quad (45)$$

From this and the equation (2), we obtain

$$\frac{\sqrt{a}}{\sqrt{2\pi}} \tilde{\phi}_{xy}(a, b) = \Phi_{x(t),y(t+b)}\left(\frac{\omega_p}{a}\right) - \frac{1}{2} \frac{w''(0)}{a^2} \Phi''_{x(t),y(t+b)}\left(\frac{\omega_p}{a}\right) + \cdots. \quad (46)$$

Here, let us consider the spectrum density estimate calculated from the wavelet transform of the corresponding cross correlation function. The equation implies that the bias of the estimate is inversely proportional to the square of the width of the wavelet basis since the width is proportional to a dilation parameter a in time domain.

This result is almost the same as the one of the Blackman-Tukey method [9], *i.e.* the bias is proportional to $1/M^2$, where M is the width of the window function. The only difference is that the window width M is constant for the Blackman-Tukey method while the width is dependent on frequency for the wavelet case.

4. Numerical Experiment

This section illustrates the above results by a numerical experiment. We estimate a power spectrum density from the wavelet transform of a auto correlation function and consider the experiment result.

To generate a test signal, a white noise of 0 (dB) was passed through the filter whose transfer function was

$$G(s) = \frac{1}{s^2 + 0.7s + 1}. \tag{47}$$

The sampling rate was set to 0.5 second and the number of samples was set to 4096. We calculated the wavelet transform of the auto-correlation function of the test signal and estimated its power spectrum density from the equation (12). The location parameter b is fixed to 0 to obtain the power spectrum density. We selected the Gabor function for a mother wavelet. It is expressed as follows.

$$\psi(t) = \exp\left(-\frac{\omega_p^2}{2\gamma^2}t^2\right)\exp(j\omega_p t) \tag{48}$$

In this experiment, ω_p and γ were set to 1(rad/sec) and 2π respectively. To simplify the calculation, the mother wavelet was multiplied by a constant so that $\psi^*(0) = 1$. This function is obtained by using the Gaussian window as follows.

$$w(t) = \exp\left(-\frac{\omega_p^2}{2\gamma^2}t^2\right) \tag{49}$$

Figure 1 shows the caluculated result. The thick line is the power spectrum estimated by the above way. The thin line is $\|G(j\omega)\|^2$, that is, the true value of the spectrum. You can see that both lines are very similar each other though there is a small bias between them. As this example, a spectrum density can be estimated from the wavelet transform of a correlation function if an appropriate mother wavelet are chosen.

Moreover, we calculated the true value of

$$\Phi_{xy}\left(\frac{\omega_p}{a}\right) - \frac{1}{2}\frac{w''(0)}{a^2}\Phi''_{xy}\left(\frac{\omega_p}{a}\right), \tag{50}$$

which is the first two terms of the right hand of the equation (46). The value is plotted by the dashed line in Fig. 1. It can be seen that the difference becomes smaller by adding the second order term.

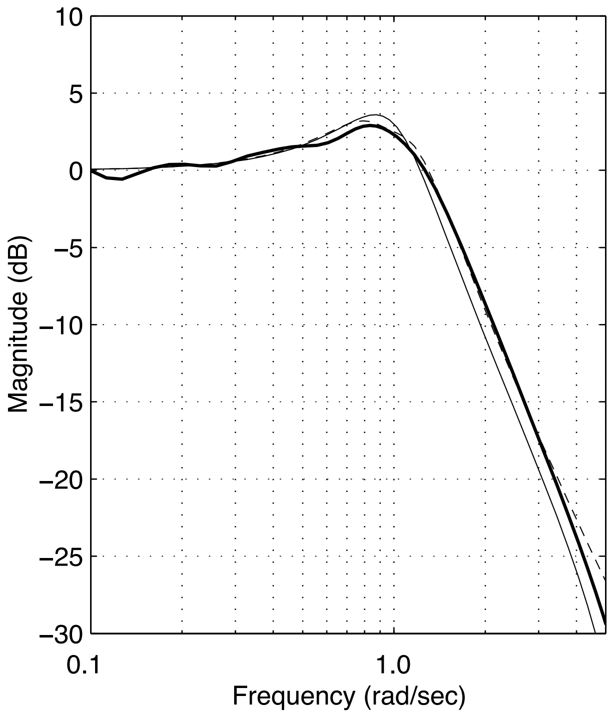

Fig. 1 Power spectrum estimation using the wavelet transform of a correlation function. The thick line is the estimated power spectrum by the wavelet transform of the auto-correlation function of a signal. It is obtained by the equation (12). The thin line is its true value. The difference between them is small. This result shows that it is possible to estimate the spectrum density by the wavelet transform of the correlation function. The dashed line is the plot of (50), which is the true value plus the first term of the estimate's bias. The difference becomes smaller.

5. Conclusion

This paper considers the relation between the wavelet transform of a correlation function and the corresponding spectrum density. It is desirable to use the windowed complex sinusoid wavelet, which is a complex sinusoid $e^{j\omega_p t}$ multiplied by a real even window function, if we expect to relate the wavelet transform of the correlation function with the spectrum density. The Gabor function is one of the commonly used wavelets of them. As mentioned in the introduction, we employed the wavelet transform of the correlation function for dead time measurement and system identification. The result theoretically justifies use of the Gabor function for these studies since they have a close relation with the spectrum density. It has great significance for the related application researches. Also for general applications using the continuous wavelet transform of the correlation function, we recommend to start their developments with such mother wavelets. They enable us to link the wavelet transform with the notion of frequency and spectrum, and it will be help to understand the connection with conventional methods. We consider that it is desirable and effective at the beginning of the development since many properties of the object are unknown initially.

There are some further researches. We have to consider the center frequency of wavelets quantitatively for general mother wavelets. It is necessary to investigate a range of the location parameter such that the discussion of this paper is still valid. The evaluation of the variance is left in spite of its importance. Another method was proposed to estimate a spectrum density based on the wavelet transform of a correlation fuction[10), 11)]. The method calculates a periodogram directly from the wavelet transform of a signal. It may be necessary to study the properties of the estimates by such a method.

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