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Nonlinear Adaptive $\mathcal{H}_\infty$ Control Systems for Bounded Variations of Parameters†

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A new class of adaptive nonlinear $\mathcal{H}_\infty$ control systems for processes with bounded variations of parameters, is proposed in this manuscript. Those control schemes are derived as solutions of particular nonlinear $\mathcal{H}_\infty$ control problems, where unknown system parameters are regarded as exogenous disturbances to the processes, and thus, in the resulting control systems, the $\mathcal{L}_2$ gains from system parameters to generalized outputs are made less than γ. The proposed control strategy can be applied to any time-varying (or time-invariant) systems, and the resulting control systems are bounded for arbitrarily large but bounded variations of time-varying parameters. Also, the control schemes are shown to be sub-optimal to some $\mathcal{H}_\infty$ cost functionals (or certain differential games), when the high-frequency gains are time-invariant.

Key Words: adaptive control, nonlinear control; $\mathcal{H}_\infty$ control, optimal control; time-varying system

1. Introduction

In the study of adaptive control, the main topics have been an asymptotic stability of adaptive control systems. So much attention has not been paid on the control performances such as transient performance and other performances [1]. On the contrary, the backstepping procedures in the last few years, have not only made it possible to analyze the stability of adaptive control systems in simpler forms, but also made it possible to discuss the transient performance of responses ($\mathcal{L}_\infty/\mathcal{L}_2$ performances) of many kinds of adaptive and nonlinear control systems [2]. Furthermore, recent researches on nonlinear $\mathcal{H}_\infty$ control and inverse optimality, could derive adaptive or nonlinear control systems which are optimal to certain meaningful cost functionals [3] [4] [5] [6] [7].

Additionally, in the past study of adaptive control, there has been another problem that the stability analysis of adaptive control systems have been focused on time-invariant processes mainly; no enough discussion for the case of time-varying systems has been made. Although several approaches have been examined for time-varying processes in the study of robust adaptive control schemes [1], those results could be applied to limited classes of time-varying systems, that is, only sufficiently small variations of time-varying parameters are accepted in those robust adaptive schemes.

The purpose of the present paper is to provide a new class of adaptive nonlinear $\mathcal{H}_\infty$ control systems for processes with bounded variations of parameters, where the control performances are discussed explicitly, and the stability analysis for time-varying systems are carried out successfully. Those control schemes are derived as solutions of particular nonlinear $\mathcal{H}_\infty$ control problems, where unknown system parameters are regarded as exogenous disturbances to the processes, and thus, in the resulting control systems, the $\mathcal{L}_2$ gains from system parameters to generalized outputs are made less than γ_i^* (the prescribed positive constant). The proposed control strategy can be applied to any time-varying (or time-invariant) systems, and the resulting control systems are bounded for arbitrarily large but bounded variations of time-varying parameters. Also, the control schemes are shown to be sub-optimal to some $\mathcal{H}_\infty$ cost functionals (or certain differential games), when the high-frequency gains are time-invariant.

2. Problem Statement and System Description

We consider the following single-input single-output nonlinear system.

$$\frac{d}{dt}e(t) = \mathcal{L}(e(t)) + \mathcal{L}(f(e,t)) + b_0 u_{fn^*-1}(t) + \Phi^T\omega_1(t) \quad (1)$$

$$u_{fi}(t) = \frac{1}{(s+\lambda)^i}u(t) \quad (\lambda > 0) \quad (2)$$

where $e(t)$ is a control variable (an output or a tracking error, *et al.*), $u(t)$ is a control input, and $f(e,t)$ is an unknown nonlinear term. $\omega_1(t)$ is a vector composed of measurable signals, b_0 and Φ are unknown system parameters

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which can be time-varying, and $\mathcal{L}(\cdot)$ is an unstructured element defined by

$$\mathcal{L}(v(t)) = G_0(s)v(t) \quad (G_0 \in \mathcal{RH}_\infty) \tag{3}$$

$\lambda\,(> 0)$ is a design parameter which is known, and n^* is a relative degree of the controlled process. The following assumptions are introduced.

Assumption 1. 1) Although b_0 can be time-varying, the sign of it remains unchanged ($b_0 > 0$ or $b_0 < 0$), and is known a priori. It is assumed that $b_0 > 0$ without loss of generality.

2) The unknown nonlinear term $f(e,t)$ is evaluated by

$$f(e,t)^2 \;\le\; f_0 \cdot \phi(e) \cdot e^2 \tag{4}$$

where f_0 is an unknown positive constant, and $\phi(e)(> 0)$ is a known function of e which is n^*-1 times differentiable with respect to its argument.

3) The magnitude of $\omega_1(t)$ is evaluated as follows [7]:

$$\|\omega_1(t)\| \;\le\; M_1 \cdot \sup_{t\ge\tau}|e(\tau)| + M_2 \quad (M_1, M_2 > 0) \tag{5}$$

Furthermore $\dot{\omega}_1(t) \sim \omega_1^{(n^*-1)}(t)$ are measurable signal vectors and are evaluated by

$$\begin{aligned}\|\omega_1^{(i)}(t)\| \;\le\; & M_3 \cdot \sup_{t\ge\tau}|e(\tau)| + M_4 \cdot \sup_{t\ge\tau}|u_{fn^*-i}(\tau)| \\ & +M_5 \\ & (M_3 \sim M_5 > 0),\ (1 \le i \le n^*-1)\end{aligned} \tag{6}$$

4) The upper bounds of the magnitudes of the nominal values Φ^*, θ_1^*, θ_2^* (time-invariant) of Φ, θ_1, θ_2 (θ_1 and θ_2 are to be introduced later, and those may be time-varying), are known a priori. Also, the upper bound of $\bar{b}_0$ and lower bound of $\underline{b}_0$ on the high-frequency gain b_0 are known such that

$$0 < \delta \le \underline{b}_0 \le b_0 \le \bar{b}_0 \le M < \infty \tag{7}$$

and the upper bound of $\bar{p}$ and lower bound of $\underline{p}$ on the parameter p are known, too.

$$p \equiv \frac{1}{b_0}, \quad \bar{p} \equiv \frac{1}{\underline{b}_0}, \quad \underline{p} \equiv \frac{1}{\bar{b}_0} \tag{8}$$

(For simplicity of notation, upper bounds are evaluated by M, and lower bounds are evaluated by δ in the manuscript, and M and δ are known.)

The control problem of this paper is to determine a control input $u(t)$ adaptively such that the overall system is stabilized for arbitrary but bounded time-varying system parameters, and additionally, the control variable $e(t)$ converges to zero asymptotically in the ideal case (stability condition), while the resulting control system becomes optimal or sub-optimal to some meaningful cost functionals (optimality condition).

Remark. The controlled process (1), (2), is shown to be the generalized form which appears in many conventional adaptive control problems. It should be noted that Assumption 1-3 is concerned with the minimum-phase property of processes

3. Nonlinear Adaptive $\mathcal{H}_\infty$ Control

The design of control systems are based on backstepping procedures [2] composed of step 1 $\sim$ step n^*, and in each steps, the control signals $v_i(t)$ are determined by applying nonlinear $\mathcal{H}_\infty$ control scheme. In the last step (**Step n***), the actual control input $u(t)$ is obtained.

Step 1) Define $z_1(t)$, $z_2(t)$ by

$$z_1(t) \;\equiv\; e(t) \tag{9}$$

$$z_2(t) \;=\; u_{fn^*-1}(t) - \alpha_1(t) \tag{10}$$

The virtual control input $\alpha_1(t)$ is determined as follows:

$$\begin{aligned}\alpha_1(t) \;&=\; -\hat{p}(t)\hat{\Phi}(t)^T\omega_1(t) + v_1(t) \\ &\equiv\; -\hat{p}(t)v_0(t) + v_1(t)\end{aligned} \tag{11}$$

where $v_1(t)$ is to be determined later based on nonlinear $\mathcal{H}_\infty$ control strategy. In this manuscript, the projection-type adaptive laws [1], where tuning parameters $\hat{\theta}$ are constrained to certain closed regions $\mathcal{S}$, are defined by

$$\begin{aligned}\dot{\hat{\theta}} \;&=\; \mathrm{Pr}(\Gamma\phi\epsilon) \\ &=\; \begin{cases} \Gamma\phi\epsilon & \text{Case I} \\ \Gamma\phi\epsilon - \Gamma\frac{\nabla g \nabla g^T}{\nabla g^T \Gamma \nabla g}\Gamma\phi\epsilon & \text{Case II}\end{cases}\end{aligned} \tag{12}$$

where

Case I : $\hat{\theta} \in \mathcal{S}^o$, or $\hat{\theta} \in \partial\mathcal{S}$ & $(\Gamma\phi\epsilon)^T\nabla g \le 0$

Case II : Otherwise

$\mathcal{S} = \{\hat{\theta} : g(\hat{\theta}) \le 0\}$

$\mathcal{S}^o$ = interior of $\mathcal{S}$, $\partial\mathcal{S}$ = boundary of $\mathcal{S}$

By utilizing those descriptions, $\hat{\theta}_1(t)$, $\hat{\theta}_2(t)$, $\hat{p}(t)$ are tuned in the following ways:

$$\dot{\hat{\theta}}_1(t) \;=\; \mathrm{Pr}\{g_{11}z_1(t)^2\} \tag{13}$$

$$\dot{\hat{\theta}}_2(t) \;=\; \mathrm{Pr}\{g_{12}\phi(z_1(t))z_1(t)^2\} \tag{14}$$

$$\dot{\hat{p}}(t) \;=\; \mathrm{Pr}\{g_{13}v_0(t)z_1(t)\} \tag{15}$$

where g_{11}, g_{12}, $g_{13} > 0$, and each constraints are given by

$$\begin{aligned} & g_{\theta_1}(\hat{\theta}_1) = \hat{\theta}_1^2 - M^2, \quad g_{\theta_2}(\hat{\theta}_2) = \hat{\theta}_2^2 - M^2 \\ & g_p(\hat{p}) = \Big(\hat{p} - \frac{\delta+M}{2}\Big)^2 - \Big(\frac{M-\delta}{2}\Big)^2 \end{aligned} \tag{16}$$

M and δ are properly selected positive constants based on Assumption 1-4. Hereafter, we are to obtain the input signal $v_1(t)$ by applying nonlinear $\mathcal{H}_\infty$ control strategy.

For this purpose, define $V_1(t)$ by

$$
\begin{aligned}
V_1(t) &= \frac{1}{2}z_1(t)^2 + \frac{1}{2}\sum_{i=1}^{2}\{\hat{\theta}_i(t) - \theta_i^*\}^2/g_{1i} \\
&\quad + \frac{\underline{b}_0}{2}\{\hat{p}(t) - \bar{p}\}^2/g_{13} \\
&\quad + \frac{1}{2}\{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\hat{\Phi}(t) - \Phi^*\} \\
&\quad + \frac{1}{2}\{\hat{b}_0(t) - \underline{b}_0\}^2/g_{15} \qquad (17)
\end{aligned}
$$

where $G_{14} = G_{14}{}^T > 0$, $g_{15} > 0$, Φ^* is a nominal value (time-invariant) of the parameter Φ (time-varying), and θ_i^* are also nominal values (time-invariant) of the parameters θ_i (time-varying) determined later. We take the time derivative of it.

$$
\begin{aligned}
\dot{V}_1(t) &\le z_1(t)\{\mathcal{L}(z_1(t)) + \mathcal{L}(f(z_1, t)) + \Phi^T\omega_1(t)\} \\
&\quad - \Phi_0^{*T}\omega_0(t)z_1(t) + (\underline{b}_0 - b_0)\hat{p}(t)v_0(t)z_1(t) \\
&\quad + \hat{\theta}_1(t)z_1(t)^2 + \hat{\theta}_2(t)\phi(z_1(t))z_1(t)^2 \\
&\quad + \{\hat{b}_0(t) + (b_0 - \underline{b}_0)\}v_1(t)z_1(t) + b_0 z_1(t)z_2(t) \\
&\quad + \{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(t) - \tau_{\phi 1}(t)\} \\
&\quad + \{\hat{b}_0(t) - \underline{b}_0\}\{\dot{\hat{b}}_0(t) - \tau_{b1}(t)\}/g_{15} \qquad (18)
\end{aligned}
$$

$$\omega_0(t) = [z_1(t),\, \phi(z_1(t))z_1(t),\, \omega_1(t)^T]^T \qquad (19)$$

$$\Phi_0^* = [\theta_1^*,\, \theta_2^*,\, \Phi^{*T}]^T \qquad (20)$$

$$\tau_{\phi 1}(t) = G_{14}\omega_1(t)z_1(t) \qquad (21)$$

$$\tau_{b1}(t) = g_{15}v_1(t)z_1(t) \qquad (22)$$

Since $\mathcal{L}(\cdot)$ is defined by (3), there exist bounded θ_{11}, θ_{12} (positive) satisfying the next inequality.

$$
\begin{aligned}
&\int_0^t z_1(\tau)\{\mathcal{L}(z_1(\tau)) + \mathcal{L}(f(z_1, \tau))\}d\tau \\
&\le \int_0^t \{\theta_{11}(\tau)z_1(\tau)^2 + \theta_{12}(\tau)\phi(z_1(\tau))z_1(\tau)^2\}d\tau + N_0 \qquad (23)
\end{aligned}
$$

where N_0 is a constant which depends on initial conditions. Then, the following relation is obtained by integrating $\dot{V}_1(t)$.

$$
\begin{aligned}
&V_1(t) - V_1(0) \\
&\le \int_0^t [\hat{\theta}_1(\tau)z_1(\tau) + \hat{\theta}_2(\tau)\phi(z_1(\tau))z_1(\tau)]z_1(\tau)d\tau \\
&\quad + \int_0^t \tilde{\Theta}_1^T\tilde{\omega}_1(\tau)z_1(\tau)d\tau \\
&\quad + \int_0^t \{\hat{b}_0(\tau) + \tilde{b}_0(\tau)\}v_1(\tau)z_1(\tau)d\tau \\
&\quad + \int_0^t b_0 z_1(\tau)z_2(\tau)d\tau \\
&\quad + \int_0^t \{\hat{\Phi}(\tau) - \Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(\tau) - \tau_{\phi 1}(\tau)\}d\tau \\
&\quad + \int_0^t \{\hat{b}_0(\tau) - \underline{b}_0\}\{\dot{\hat{b}}_0(\tau) - \tau_{b1}(\tau)\}d\tau/g_{15} + N_0 \qquad (24)
\end{aligned}
$$

$$\Phi_{01} \equiv [\theta_{11},\, \theta_{12},\, \Phi^T]^T \qquad (25)$$

$$\tilde{\Theta}_1 \equiv [(\Phi_{01} - \Phi_0^*)^T,\, \underline{b}_0 - b_0]^T \qquad (26)$$

$$\tilde{b}_0 \equiv b_0 - \underline{b}_0 \ (\ge 0) \qquad (27)$$

$$\tilde{\omega}_1(t) \equiv [\omega_0(t)^T,\, \hat{p}(t)v_0(t)\}]^T \qquad (28)$$

From that evaluation of $V_1(t)$, we introduce the following virtual process

$$
\begin{aligned}
\dot{z}_1 &= \hat{\theta}_1 z_1 + \hat{\theta}_2\phi(z_1)z_1 + \tilde{\omega}_1^T\tilde{\Theta}_1 + (\hat{b}_0 + \tilde{b}_0)v_1 \\
&\equiv f_1(z_1) + g_{11}d_1 + g_{12}v_1 \qquad (29) \\
&f_1(z_1) = \hat{\theta}_1 z_1 + \hat{\theta}_2\phi(z_1)z_1 \\
&g_{11} = \tilde{\omega}_1^T, \ \ d_1 = \tilde{\Theta}_1, \ \ g_{12} = \hat{b}_0 + \tilde{b}_0 \qquad (30)
\end{aligned}
$$

and stabilize it via v_1 by utilizing nonlinear $\mathcal{H}_\infty$ control strategy, where unknown parameters $\tilde{\Theta}_1$ are regarded as exogenous disturbances to the process. For this purpose, consider the Hamilton-Jacobi-Isaacs equation

$$
\begin{aligned}
&\frac{\partial\tilde{V}_1}{\partial z_1}f_1 + \frac{1}{4}\left(\frac{\|g_{11}\|^2}{\gamma_1^{*2}} - \frac{g_{12}^2}{r_1}\right)\left(\frac{\partial\tilde{V}_1}{\partial z_1}\right)^2 \\
&\qquad + h_1 z_1^2 \le 0 \qquad (31)
\end{aligned}
$$

where the solution $\tilde{V}_1$ is given by the next equation

$$\tilde{V}_1(t) = \frac{1}{2}z_1(t)^2 \qquad (32)$$

h_1 and r_1 are positive functions to be determined from the inequality (31) based on inverse optimality for the given solution $\tilde{V}_1$ (32) and the positive constant γ_1^*. The substitution of (32) into (31) yields

$$
\begin{aligned}
&\hat{\theta}_1 z_1^2 + \hat{\theta}_2\phi(z_1)z_1^2 + \left\{\frac{\|\tilde{\omega}_1\|^2}{\gamma_1^{*2}} - \frac{(\hat{b}_0 + \tilde{b}_0)^2}{r_1}\right\}\frac{z_1^2}{4} \\
&\qquad + h_1 z_1^2 \le 0 \qquad (33)
\end{aligned}
$$

Since the unknown element $\tilde{b}_0(\ge 0)$ is included in the above inequality, we are to obtain h_1 and r_1 satisfying the next relation, which is a sufficient condition for the original inequality (33)

$$\hat{\theta}_1 z_1^2 + \hat{\theta}_2\phi(z_1)z_1^2 + \left(\frac{\|\tilde{\omega}_1\|^2}{\gamma_1^{*2}} - \frac{\hat{b}_0^2}{r_1}\right)\frac{z_1^2}{4} + h_1 z_1^2 \le 0 \qquad (34)$$

From that, the control signal v_1 is derived as a solution for the nonlinear $\mathcal{H}_\infty$ problem.

$$v_1^* = -\frac{1}{2r_1}g_{12}\frac{\partial\tilde{V}_1}{\partial z_1} = -\frac{1}{2r_1}(\hat{b}_0 + \tilde{b}_0)z_1 \qquad (35)$$

Since the unknown element $\tilde{b}_0$ is included in the above equation, the actual input signal is replaced by

$$v_1^* = -\frac{\hat{b}_0}{2r_1} z_1 \tag{36}$$

Then, we obtain the following relation for the original process (1), (2), $V_1(t)$ (17), and the input signal $v_1(t)$ which is not necessarily equal to $v_1^*(t)$ (35).

$$\begin{aligned}
V_1(t) - V_1(0) \le & \int_0^t r_1 \left\{ \frac{\hat{b}_0(\tau)z_1(\tau)}{2r_1} + v_1(\tau) \right\}^2 d\tau \\
& - \int_0^t \{h_1 z_1(\tau)^2 + r_1 v_1(\tau)^2\} d\tau \\
& - \int_0^t \gamma_1^{*2} \left\| \tilde{\Theta}_1(\tau) - \frac{1}{2\gamma_1^{*2}} \tilde{\omega}_1(\tau) z_1(\tau) \right\|^2 d\tau \\
& + \gamma_1^{*2} \int_0^t \|\tilde{\Theta}_1(\tau)\|^2 d\tau \\
& + \int_0^t \tilde{b}_0(\tau) v_1(\tau) z_1(\tau) d\tau + \int_0^t b_0 z_1(\tau) z_2(\tau) d\tau \\
& + \int_0^t \{\hat{\Phi}(\tau) - \Phi^*\}^T G_{14}^{-1} \{\dot{\hat{\Phi}}(\tau) - \tau_{\phi 1}(\tau)\} d\tau \\
& + \int_0^t \{\hat{b}_0(\tau) - \underline{b}_0\}\{\dot{\hat{b}}_0(\tau) - \tau_{b1}(\tau)\} d\tau / g_{15} + N_0
\end{aligned} \tag{37}$$

<u>**Step i)**</u> $(2 \le i \le n^*)$ Take the time derivative of $z_i(t)$.

$$z_i(t) \equiv u_{fn^*-i+1}(t) - \alpha_{i-1}(t) \tag{38}$$

$$\begin{aligned}
\dot{z}_i(t) = & -\lambda u_{fn^*-i+1}(t) + u_{fn^*-i}(t) - \beta_{i-1}(t) \\
& -\gamma_{i-1}(t)\{\mathcal{L}(z_1(t)) + \mathcal{L}(f(z_1, t)) \\
& + b_0 u_{fn^*-1}(t) + \Phi^T \omega_1(t)\} \\
& -\gamma_{Ki-1}(t) \mathrm{Pr}\{G_1 \tilde{v}_1(t) z_1(t)\} \\
& -\gamma_{\phi i-1}(t) \dot{\hat{\Phi}}(t) - \gamma_{bi-1}(t) \dot{\hat{b}}_0(t)
\end{aligned} \tag{39}$$

$$\begin{aligned}
\beta_{i-1}(t) = & \frac{\partial \alpha_{i-1}}{\partial \omega_{i-1}} \dot{\omega}_{i-1}(t) \\
& + \sum_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_j} \{-\lambda u_{fn^*-j+1}(t) \\
& + u_{fn^*-j}(t) - \beta_{j-1}(t)\}
\end{aligned} \tag{40}$$

$$\gamma_{i-1}(t) = \frac{\partial \alpha_{i-1}}{\partial z_1} - \sum_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_j} \gamma_{j-1}(t) \tag{41}$$

$$\gamma_{Ki-1}(t) = \frac{\partial \alpha_{i-1}}{\partial \hat{K}_1} - \sum_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_j} \gamma_{Kj-1}(t) \tag{42}$$

$$\gamma_{\phi i-1}(t) = \frac{\partial \alpha_{i-1}}{\partial \hat{\Phi}} - \sum_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_j} \gamma_{\phi j-1}(t) \tag{43}$$

$$\gamma_{bi-1}(t) = \frac{\partial \alpha_{i-1}}{\partial \hat{b}_0} - \sum_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_j} \gamma_{bj-1}(t) \tag{44}$$

$$\hat{K}_1(t) = [\hat{\theta}_1(t),\, \hat{\theta}_2(t),\, \hat{p}(t)]^T \tag{45}$$

$$\tilde{v}_1(t) = [z_1(t),\, \phi(z_1(t)) z_1(t),\, v_0(t)]^T \tag{46}$$

$$G_1 = \mathrm{diag}(g_{11},\, g_{12},\, g_{13}) \tag{47}$$

$$\begin{aligned}
\omega_{i-1}(t) = & \text{ vector signals composed of elements} \\
& \{u_{fn^*-1}(t) \sim u_{fn^*-i+2},\, \omega_{i-2}(t),\, \dot{\omega}_{i-2}(t)\} \\
& (3 \le i \le n^*)
\end{aligned} \tag{48}$$

For $z_i(t)$, we introduce $z_{i+1}(t)$ and determine the virtual control $\alpha_i(t)$ so as to stabilize $z_i(t)$.

$$z_{i+1}(t) \equiv u_{fn^*-i}(t) - \alpha_i(t) \tag{49}$$

$$\begin{aligned}
\alpha_i(t) = & \lambda u_{fn^*-i+1}(t) + \beta_{i-1}(t) - \hat{c}_i(t) z_{i-1}(t) \\
& + \gamma_{i-1}(t) \hat{\Phi}(t)^T \omega_1(t) \\
& + \hat{b}_0(t) \gamma_{i-1}(t) u_{fn^*-1}(t) + v_i(t) + \tilde{\alpha}_i(t)
\end{aligned} \tag{50}$$

$$\hat{c}_i(t) = \begin{cases} \hat{b}_0(t) & (i = 2) \\ 1 & (i \ge 3) \end{cases} \tag{51}$$

where $\tilde{\alpha}_i(t)$ is an auxiliary signal to be determined later, and the input signal $v_i(t)$ is to be obtained by applying nonlinear $\mathcal{H}_\infty$ control strategy similar to **Step 1**. For this purpose, we define $V_i(t)$ by

$$V_i(t) = \frac{1}{2} z_i(t)^2 \tag{52}$$

The next inequalities hold for certain positive functions θ_{21}, θ_{22}, and arbitrary positive constants k_{i1}, k_{i2}.

$$\begin{aligned}
& - \int_0^t \gamma_{i-1}(\tau)\{\mathcal{L}(z_1(\tau) + \mathcal{L}(f(z_1, \tau))\} z_i(\tau) d\tau \\
& \le k_{i1} \int_0^t \gamma_{i-1}(\tau)^2 z_i(\tau)^2 d\tau + \frac{1}{k_{i1}} \int_0^t \{\theta_{21}(\tau) z_1(\tau)^2 \\
& \quad + \theta_{22}(\tau)^2 \phi(z_1(\tau)) z_1(\tau)^2\} d\tau + N_0
\end{aligned} \tag{53}$$

$$\begin{aligned}
& - \int_0^t \gamma_{Ki-1}(\tau) \mathrm{Pr}\{G_1 \tilde{v}_1(\tau) z_1(\tau)\} z_i(\tau) d\tau \\
& \le \int_0^t \|\gamma_{Ki-1}(\tau)\| \|G_1\| \|\tilde{v}_1(\tau)\| |z_1(\tau)| |z_i(\tau)| d\tau \\
& \le k_{i2} \int_0^t \|\gamma_{Ki-1}(\tau)\|^2 \|\tilde{v}_1(\tau)\|^2 z_i(\tau)^2 d\tau \\
& \quad + \frac{1}{4k_{i2}} \|G_1\|^2 \int_0^t z_1(\tau)^2 d\tau
\end{aligned} \tag{54}$$

We take the time derivative of $V_i(t)$ and integrate it again. Then, similar to the previous steps, we get the following inequality.

$$\begin{aligned}
& \int_0^t c_i z_{i-1}(\tau) z_i(\tau) d\tau \\
& \quad + \int_0^t \{\hat{\Phi}(\tau) - \Phi^*\}^T G_{14}^{-1} \{\dot{\hat{\Phi}}(\tau) - \tau_{\phi i-1}(\tau)\} d\tau \\
& \quad + \int_0^t \{\hat{b}_0(\tau) - b_0^*\}\{\dot{\hat{b}}_0(\tau) - \tau_{bi-1}(\tau)\} d\tau / g_{15} \\
& \quad + \int_0^t \dot{V}_i(\tau) d\tau \\
& \le \int_0^t z_i(\tau) z_{i+1}(\tau) d\tau + k_{i1} \int_0^t \gamma_{i-1}(\tau)^2 z_i(\tau)^2 d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{k_{i1}}\int_0^t \{\theta_{21}(\tau)z_1(\tau)^2+\theta_{22}(\tau)\phi(z_1(\tau))z_1(\tau)^2\}d\tau\\
&+k_{i2}\int_0^t \|\gamma_{Ki-1}(\tau)\|^2\|\tilde{v}_1(\tau)\|^2 z_i(\tau)^2 d\tau\\
&+\frac{1}{4k_{i2}}\|G_1\|^2\int_0^t z_1(\tau)^2 d\tau+\int_0^t \tilde{\Theta}_2(\tau)^T\tilde{\omega}_i(\tau)z_i(\tau)d\tau\\
&+\int_0^t \{\hat{\Phi}(\tau)-\Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(\tau)-\tau_{\phi i}(\tau)\}d\tau\\
&+\int_0^t \{\hat{b}_0(\tau)-b_0^*\}\{\dot{\hat{b}}_0(\tau)-\tau_{bi}(\tau)\}d\tau/g_{15}\\
&-\int_0^t \{\gamma_{\phi i-1}(\tau)\dot{\hat{\Phi}}(\tau)+\gamma_{bi-1}(\tau)\dot{\hat{b}}_0(\tau)\}z_i(\tau)d\tau\\
&+\int_0^t z_i(\tau)\{v_i(\tau)+\tilde{\alpha}_i(\tau)\}d\tau+N_0 \qquad (55)
\end{aligned}
$$

$$\tilde{\Theta}_2=[b_0-\underline{b}_0,\,(\Phi-\Phi^*)^T]^T \qquad (56)$$

$$\tilde{\omega}_2(t)=[z_1(t)-\gamma_1(t)u_{fn^*-1}(t),\,-\gamma_1(t)\omega_1(t)^T]^T \qquad (57)$$

$$\tilde{\omega}_i(t)=[-\gamma_{i-1}(t)u_{fn^*-1}(t),\,-\gamma_{i-1}(t)\omega_1(t)^T]^T \quad (i\ge 3) \qquad (58)$$

$$\tau_{\phi i}(t)=\tau_{\phi i-1}(t)-G_{14}\gamma_{i-1}(t)\omega_1(t)z_i(t) \qquad (59)$$

$$\tau_{b2}(t)=\tau_{b1}(t)+g_{15}\{z_1(t)-\gamma_1(t)u_{fn^*-1}(t)\}z_2(t) \qquad (60)$$

$$\tau_{bi}(t)=\tau_{bi-1}(t)-g_{15}\gamma_{i-1}(t)u_{fn^*-1}(t)z_i(t) \quad (i\ge 3) \qquad (61)$$

$$c_i=\begin{cases} b_0 & (i=2)\\ 1 & (i\ge 3)\end{cases} \qquad (62)$$

From that relation, we introduce the virtual process

$$
\begin{aligned}
\dot{z}_i &= k_{i1}\gamma_{i-1}^2 z_i+k_{i2}\|\gamma_{Ki-1}\|^2\|\tilde{v}_1\|^2 z_i+\tilde{\omega}_i^T\tilde{\Theta}_2+v_i\\
&\equiv f_i(z_1,\cdots,z_i)+g_{i1}d_i+g_{i2}v_i \qquad (63)\\
&f_i(z_1,\cdots,z_i)=k_{i1}\gamma_{i-1}^2 z_i+k_{i2}\|\gamma_{Ki-1}\|^2\|\tilde{v}_1\|^2 z_i,\\
&g_{i1}=\tilde{\omega}_i^T,\ \ d_i=\tilde{\Theta}_2,\ \ g_{i2}=1 \qquad (64)
\end{aligned}
$$

and stabilize it via v_i by applying nonlinear $\mathcal{H}_\infty$ control strategy, where unknown parameters $\tilde{\Theta}_2$ are regarded as exogenous disturbances to the process. For this purpose, consider the following Hamilton-Jacobi-Isaacs equation

$$\frac{\partial\tilde{V}_i}{\partial z_i}f_i+\frac{1}{4}\left(\frac{\|g_{i1}\|^2}{\gamma_i^{*2}}-\frac{g_{i2}^2}{r_i}\right)\left(\frac{\partial\tilde{V}_i}{\partial z_i}\right)^2+h_i z_i^2\le 0 \qquad (65)$$

where the solution $\tilde{V}_i$ is given by the next equation

$$\tilde{V}_i(t)=V_i(t)=\frac{1}{2}z_i(t)^2 \qquad (66)$$

h_i and r_i are positive functions to be determined from the inequality (65) based on inverse optimality for the given solution $\tilde{V}_i$ (66) and the positive constant γ_i^*. The substitution of (66) into (65) yields

$$
\begin{aligned}
&k_{i1}\gamma_{i-1}^2 z_i^2+k_{i2}\|\gamma_{Ki-1}\|^2\|\tilde{v}_1\|^2 z_i^2\\
&\quad+\left\{\frac{\|\tilde{\omega}_i\|^2}{\gamma_i^{*2}}-\frac{1}{r_i}\right\}\frac{z_i^2}{4}+h_i z_i^2\le 0 \qquad (67)
\end{aligned}
$$

From that the control input $v_i(t)$ are obtained as solutions for nonlinear $\mathcal{H}_\infty$ control problems as follows:

$$v_i^*=-\frac{1}{2r_i}g_{i2}\frac{\partial\tilde{V}_i}{\partial z_i}=-\frac{1}{2r_i}z_i \qquad (68)$$

Then, we derive the following relation for the original process (1), (2), $V_1(t)\sim V_i(t)$ ((17), (52)) and the input signals $v_1(t)\sim v_i(t)$ which are not necessarily equal to $v_1^*(t)\sim v_i^*(t)$ ((35), (68)).

$$
\begin{aligned}
&\sum_{j=1}^{i}\{V_j(t)-V_j(0)\}\le\int_0^t r_1\left\{\frac{\hat{b}_0(\tau)z_1(\tau)}{2r_1}+v_1(\tau)\right\}^2 d\tau\\
&\quad-\int_0^t\{h_1 z_1(\tau)^2+r_1 v_1(\tau)^2\}d\tau\\
&\quad-\int_0^t\gamma_1^{*2}\left\|\tilde{\Theta}_{1i}(\tau)-\frac{1}{2\gamma_1^{*2}}\tilde{\omega}_1(\tau)z_1(\tau)\right\|^2 d\tau\\
&\quad+\gamma_1^{*2}\int_0^t\|\tilde{\Theta}_{1i}(\tau)\|^2 d\tau\\
&\quad+\int_0^t\tilde{b}_0(\tau)v_1(\tau)z_1(\tau)d\tau\\
&\quad+\sum_{j=2}^{i}\left[\int_0^t r_j\left\{\frac{z_j(\tau)}{2r_j}+v_j(\tau)\right\}^2 d\tau\right.\\
&\quad-\int_0^t\{h_j z_j(\tau)^2+r_j v_j(\tau)^2\}d\tau\\
&\quad-\int_0^t\gamma_j^{*2}\left\|\tilde{\Theta}_2(\tau)-\frac{1}{2\gamma_j^{*2}}\tilde{\omega}_j(\tau)z_j(\tau)\right\|^2 d\tau\\
&\quad\left.+\gamma_j^{*2}\int_0^t\|\tilde{\Theta}_2(\tau)\|^2 d\tau\right]\\
&\quad+\int_0^t z_i(\tau)z_{i+1}(\tau)d\tau\\
&\quad+\int_0^t\{\hat{\Phi}(\tau)-\Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(\tau)-\tau_{\phi i}(\tau)\}d\tau\\
&\quad+\int_0^t\{\hat{b}_0(\tau)-\underline{b}_0\}\{\dot{\hat{b}}_0(\tau)-\tau_{bi}(\tau)\}d\tau/g_{15}\\
&\quad-\sum_{j=2}^{i}\int_0^t\gamma_{\phi j-1}(\tau)\dot{\hat{\Phi}}(\tau)z_j(\tau)d\tau\\
&\quad-\sum_{j=2}^{i}\int_0^t\gamma_{bj-1}(\tau)\dot{\hat{b}}_0(\tau)z_j(\tau)d\tau\\
&\quad+\sum_{j=2}^{i}\int_0^t\tilde{\alpha}_j(\tau)z_j(\tau)d\tau+N_0 \qquad (69)
\end{aligned}
$$

$$\tilde{\Theta}_{1i}=[(\Phi_{0i}-\Phi_0^*)^T,\,\underline{b}_0-b_0]^T \qquad (70)$$

$$\Phi_{0i}=\left[\theta_{11}+\sum_{j=2}^{i}\left(\frac{\theta_{21}}{k_{j1}}+\frac{\|G_1\|^2}{4k_{j2}}\right),\ \theta_{12}+\sum_{j=2}^{i}\frac{\theta_{22}}{k_{j1}},\,\Phi^T\right]^T \qquad (71)$$

In the last **Step** n^***)**, the actual control input is obtained as

$$u(t) = \alpha_{n^*}(t) \tag{72}$$

Step $\mathbf{n^*+1}$) The tuning laws of $\hat{\Phi}(t)$, $\hat{b}_0(t)$ and the auxiliary signals $\tilde{\alpha}_i(t)$ are determined such that

$$\dot{\hat{\Phi}}(t) = \Pr\{\tau_{\phi n^*}(t)\} \tag{73}$$

$$\dot{\hat{b}}_0(t) = \Pr\{\tau_{bn^*}(t)\} \tag{74}$$

$$\begin{aligned}\tilde{\alpha}_i(t) = &-k_{i3}\|\gamma_{\phi i-1}(t)\|^2\|\omega_1(t)\|^2 z_i(t)\\ &-(n^*-1)k_{i4}\|\gamma_{\phi i-1}(t)\|^2\|\omega_1(t)\|^2 z_i(t)\\ &-\sum_{j=2}^{n^*}\frac{\|G_{14}\|^2}{4k_{j4}}\gamma_{i-1}(t)^2 z_i(t) - k_{i5}\gamma_{bi-1}(t)^2 v_1(t)^2 z_i(t)\\ &-k_{i6}\gamma_{bi-1}(t)^2 z_2(t)^2 z_i(t)\\ &-(n^*-1)k_{i7}\gamma_{bi-1}(t)^2 u_{fn^*-1}(t)^2 z_i(t)\\ &-\sum_{j=2}^{n^*}\frac{g_{15}^2}{4k_{j7}}\gamma_{i-1}(t)^2 z_i(t) \qquad (2\le i\le n^*)\end{aligned} \tag{75}$$

where the constraints are defined by

$$g_\phi(\hat{\Phi}) = \|\hat{\Phi}\|^2 - M^2, \quad g_b(\hat{b}_0) = \|\hat{b}_0\|^2 - M^2 \tag{76}$$

$M(>0)$ is determined similarly to (16). Then, the following inequality is derived by utilizing the property of projection type adaptive laws.

$$\begin{aligned}&\{\hat{\Phi}(t)-\Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(t)-\tau_{\phi n^*}(t)\}\\ &\quad+\{\hat{b}_0(t)-b_0^*\}\{\dot{\hat{b}}_0(t)-\tau_{bn^*}(t)\}d\tau/g_{15}\\ &=\{\hat{\Phi}(t)-\Phi^*\}^T G_{14}^{-1}\{\Pr(\tau_{\phi n^*}(t))-\tau_{\phi n^*}(t)\}\\ &\quad+\{\hat{b}_0(t)-b_0^*\}\{\Pr(\tau_{bn^*}(t))-\tau_{bn^*}(t)\}d\tau/g_{15}\le 0\end{aligned} \tag{77}$$

Also, from the evaluation of $\Pr(\cdot)$ (12), it follows that

$$\|\Pr(\Gamma\phi\epsilon)\| \le \|\Gamma\phi\epsilon\| \tag{78}$$

and the next relation is obtained.

$$\begin{aligned}&-\sum_{j=2}^{n^*}\gamma_{\phi j-1}(t)\dot{\hat{\Phi}}(t)z_j(t) - \sum_{j=3}^{n^*}\gamma_{bj-1}(t)\dot{\hat{b}}_0(t)z_j(t)\\ &\quad+\sum_{j=2}^{n^*}\tilde{\alpha}_j(t)z_j(t)\\ &\le\sum_{j=2}^{n^*}\frac{\|G_{14}\|^2}{4k_{j3}}z_1(t)^2+\sum_{j=2}^{n^*}\frac{g_{15}^2}{4k_{j5}}z_1(t)^2\\ &\quad+\sum_{j=2}^{n^*}\frac{g_{15}^2}{4k_{j6}}z_1(t)^2\end{aligned} \tag{79}$$

Finally, we derive the following evaluation of $V_i(t)$ ((17), (52)) by utilizing (77), (78), (79), where $v_i(t)$ are not necessarily equal to $v_i^*(t)$((35), (68)).

$$\begin{aligned}\sum_{i=1}^{n^*}\{V_i(t)-V_i(0)\} \le &\int_0^t r_1\left\{\frac{\hat{b}_0(\tau)z_1(\tau)}{2r_1}+v_1(\tau)\right\}^2 d\tau\\ &-\int_0^t\{h_1 z_1(\tau)^2+r_1 v_1(\tau)^2\}d\tau\\ &-\int_0^t\gamma_1^{*2}\left\|\tilde{\Theta}_{1n^*}(\tau)-\frac{1}{2\gamma_1^{*2}}\tilde{\omega}_1(\tau)z_1(\tau)\right\|^2 d\tau\\ &+\gamma_1^{*2}\int_0^t\|\tilde{\Theta}_{1n^*}(\tau)\|^2 d\tau+\int_0^t\tilde{b}_0(\tau)v_1(\tau)z_1(\tau)d\tau\\ &+\sum_{i=2}^{n^*}\left[\int_0^t r_i\left\{\frac{z_i(\tau)}{2r_i}+v_i(\tau)\right\}^2 d\tau\right.\\ &-\int_0^t\{h_i z_i(\tau)^2+r_i v_i(\tau)^2\}d\tau\\ &-\int_0^t\gamma_i^{*2}\left\|\tilde{\Theta}_2(\tau)-\frac{1}{2\gamma_i^{*2}}\tilde{\omega}_i(\tau)z_i(\tau)\right\|^2 d\tau\\ &\left.+\gamma_i^{*2}\int_0^t\|\tilde{\Theta}_2(\tau)\|^2 d\tau\right]+N_0\end{aligned} \tag{80}$$

$$\tilde{\Theta}_{1n^*} = [(\Phi_{0n^*}-\Phi_0^*)^T,\ \underline{b}_0 - b_0]^T \tag{81}$$

$$\Phi_{0n^*} = \left[\theta_1,\ \theta_2,\ \Phi^T\right]^T \tag{82}$$

$$\begin{aligned}\theta_1 = \theta_{11}+\sum_{j=2}^{n^*}\Bigg(&\frac{\theta_{21}}{k_{j1}}+\frac{\|G_1\|^2}{4k_{j2}}+\frac{\|G_{14}\|^2}{4k_{j3}}\\ &+\frac{g_{15}^2}{4k_{j5}}+\frac{g_{15}^2}{4k_{j6}}\Bigg)\end{aligned} \tag{83}$$

$$\theta_2 = \theta_{12}+\sum_{j=2}^{n^*}\frac{\theta_{22}}{k_{j1}} \tag{84}$$

θ_1^*, θ_2^* in (17), are nominal values (time-invariant) of θ_1, θ_2 in (83), (84).

Then, we have the following main theorems.

Theorem 1. The adaptive control system described above (where $v_1^*(t) \sim v_{n^*}^*(t)$ ((35), (68)) are included) is uniformly bounded for arbitrary bounded variation of system parameters b_0, p, Φ, θ_{11}, θ_{12}, θ_{21}, θ_{22}.

Proof. By introducing state variables $w(t)$ of the stable systems (state-space representation (F,G)), the unstructured elements $\mathcal{L}(z_1(t))$ and $\mathcal{L}(f(z_1,t))$ are written in the following:

$$\dot{w}(t) = Fw(t)+G\begin{bmatrix} z_1(t)\\ \phi(z_1(t))z_1(t)\end{bmatrix} \tag{85}$$

$$\begin{aligned}&\|\mathcal{L}(z_1(t))\|^2+\|\mathcal{L}(f(y,t))\|^2\\ &\quad\le M_1\|w(t)\|^2+M_2 z_1(t)^2+M_3\phi(z_1(t))z_1(t)^2\end{aligned} \tag{86}$$

$$PF+F^TP=-I \quad (P=P^T>0) \tag{87}$$

Adding $w(t)$, $\bar{V}(t)$ is defined by

$$\bar{V}(t) = \frac{1}{2}\sum_{i=1}^{n^*}z_i(t)^2+lw(t)^TPw(t) \quad (l>0) \tag{88}$$

We take the time derivative of $\tilde{V}(t)$, then we have the following evaluation for properly selected l, δ_0, δ_1, D^*.

$$\begin{aligned}\dot{\bar{V}}(t) &\le -\sum_{i=1}^{n^*}\{h_i z_i(t)^2 + r_i v_i(t)^2\} - \frac{\hat{b}_0(t)\tilde{b}_0(t)}{2r_1} z_1(t)^2 \\ &\quad + \max_i\{\gamma_i^{*2}\}\cdot D^* - \delta_1\|w(t)\|^2 \\ &\le -\delta_0\bar{V}(t) + \max_i\{\gamma_i^{*2}\}\cdot D^* \end{aligned} \quad (89)$$

where $0 < \delta_0$, $D^* < \infty$. Hence, we show that the adaptive system remains bounded for any bounded variations of system parameters.

Theorem 1 holds for arbitrary bounded tuning parameters which are not necessarily determined by the adaptive laws (13), (14), (15), (73), (74) stated in the manuscript. On the contrary, hereafter, we are to utilize the projection-type adaptive laws (13), (14), (15), (73), (74).

Theorem 2. For that adaptive control systems (including $v_1 \sim v_{n^*}$), we assume that b_0 is time-invariant. Then, $v_1^* \sim v_{n^*}^*$ ((35), (68)) are sub-optimal control inputs which minimize the upper bound on the following cost functional.

$$\begin{aligned}J(t) \equiv \sup_{\tilde{\Theta}_{1n^*},\tilde{\Theta}_2\in\mathcal{L}_2}&\Bigg[\sum_{i=1}^{n^*}\int_0^t\{h_i z_i(\tau)^2 + r_i v_i(\tau)^2\}d\tau \\ &+\sum_{i=1}^{n^*}V_i(t) - \gamma_1^{*2}\int_0^t\|\tilde{\Theta}_{1n^*}(\tau)\|^2 d\tau \\ &-\sum_{i=2}^{n^*}\gamma_i^{*2}\int_0^t\|\tilde{\Theta}_2(\tau)\|^2 d\tau\Bigg]\end{aligned} \quad (90)$$

Also we have the next inequality for those sub-optimal $v_1^* \sim v_{n^*}^*$.

$$\begin{aligned}&\sum_{i=1}^{n^*}\int_0^t\{h_i z_i(\tau)^2 + r_i v_i(\tau)^2\}d\tau + \sum_{i=1}^{n^*}V_i(t) \\ &\quad\le \sum_{i=1}^{n^*}V_i(0) + \gamma_1^{*2}\int_0^t\|\tilde{\Theta}_{1n^*}(\tau)\|^2 d\tau \\ &\qquad+\sum_{i=2}^{n^*}\gamma_i^{*2}\int_0^t\|\tilde{\Theta}_2(\tau)\|^2 d\tau + N_0\end{aligned} \quad (91)$$

Especially, if $\tilde{\Theta}_{1n^*}, \tilde{\Theta}_2 \in \mathcal{L}_2$, then it holds that $z_1(t) \sim z_{n^*}(t) \to 0$.

Theorem 3. For that adaptive system (including $v_1^* \sim v_{n^*}^*$ ((35), (68))), we assume that b_0 is not time-invariant. Then, the following inequality is derived.

$$\begin{aligned}&\sum_{i=1}^{n^*}\int_0^t\{h_i z_i(\tau)^2 + r_i v_i(\tau)^2\}d\tau + \sum_{i=1}^{n^*}V_i(t) \\ &\quad+\int_0^t\frac{\tilde{b}_0(\tau)\hat{b}_0(\tau)}{2r_1}z_1(\tau)^2 d\tau \\ &\quad\le \sum_{i=1}^{n^*}V_i(0) + \gamma_1^{*2}\int_0^t\|\tilde{\Theta}_{1n^*}(\tau)\|^2 d\tau \\ &\qquad+\sum_{i=2}^{n^*}\gamma_i^{*2}\int_0^t\|\tilde{\Theta}_2(\tau)\|^2 d\tau + N_0\end{aligned} \quad (92)$$

where $\frac{\tilde{b}_0(t)\hat{b}_0(t)}{2r_1}z_1(t)^2 \ge 0$. Especially, if $\tilde{\Theta}_{1n^*}, \tilde{\Theta}_2 \in \mathcal{L}_2$, then, it holds that $z_1(t) \sim z_{n^*}(t) \to 0$, and the adaptive system converges to the sub-optimal control system described in Theorem 2.

Proof. From the evaluation of $\sum_{i=1}^{n^*}V_i(t)$ (80), Theorem 2 and 3 are easily derived.

Theorem 4. Assume that $(\Phi_{0n^*} - \Phi_0^*)$, $(b_0 - b_0^*)$, $(p - p^*) \in \mathcal{L}_2$ and $b_0 \to b_0^*$, $p \to p^*$ for certain constant Φ_0^*, b_0^*, p^* ($b_0^* p^* = 1$). Then, we have $z_1(t) \sim z_{n^*}(t) \to 0$ (as $t \to \infty$).

Proof. The positive function $V_1(t)$ is newly redefined by

$$\begin{aligned}V_1(t) &= \frac{1}{2}z_1(t)^2 + \frac{1}{2}\sum_{i=1}^{2}\{\hat{\theta}_i(t) - \theta_i^*\}^2/g_{1i} \\ &\quad+\frac{b_0^*}{2}\{\hat{p}(t) - p^*\}^2/g_{13} \\ &\quad+\frac{1}{2}\{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\hat{\Phi}(t) - \Phi^*\} \\ &\quad+\frac{1}{2}\{\hat{b}_0(t) - b_0^*\}^2/g_{15}\end{aligned} \quad (93)$$

Then, we have the similar evaluation of $\sum_{i=1}^{n^*}V_i(t)$, where $\tilde{\Theta}_{1n^*}(t)$ and $\tilde{b}_0(t)$ are differently defined by

$$\tilde{\Theta}_{1n^*}(t) \equiv [(\Phi_{0n^*} - \Phi_0^*)^T, b_0^* - b_0]^T \quad (94)$$

$$\tilde{b}_0(t) \equiv b_0 - b_0^* \quad (95)$$

Contrary to the previous $\tilde{b}_0(t)$ (27), for this new $\tilde{b}_0(t)$, it does not hold that $\tilde{b}_0(t) \ge 0$. However, since $\tilde{b}_0(t)$ converges to zero asymptotically, it holds that

$$h_1 z_1(t)^2 + \frac{\tilde{b}_0(t)\hat{b}_0(t)}{2r_1}z_1(t)^2 \ge \delta_0 z_1(t)^2 \quad (\forall t \ge T), \quad (\delta_0 > 0) \quad (96)$$

for sufficiently large $T > 0$. Then, rewriting the inequality (92) with the initial time T, we show that $z_1(t) \sim z_{n^*}(t) \to 0$, where the boundedness of the adaptive system is also considered.

Up to now, the general forms of the control schemes were provided by (34), (67), (35), (68), Theorem 1 ∼ Theorem 4. Next, h_i and r_i are solved, and the explicit descriptions of the control schemes are given by assuming specified forms to h_i and r_i.

Solution I. From (34), (67), r_1, r_i ($i \ge 2$) can be chosen such that

$$r_1 = \frac{r_{10}}{k_{i8} + k_{i9}\|\tilde{\omega}_1\|^2 + k_{110}\{\hat{\theta}_1 + \hat{\theta}_2\phi(z_1)\}}$$

$$r_i = \frac{r_{i0}}{k_{i8} + k_{i9}\|\tilde{\omega}_i\|^2 + k_{i10}\{k_{i1}\gamma_{i-1}^2 + k_{i2}\|\gamma_{Ki-1}\|^2\|\tilde{v}_1\|^2\}}$$

$$(k_{18},\, k_{19},\, k_{110},\, r_{10},\, k_{i8},\, k_{i9},\, k_{i10},\, r_{i0} > 0) \qquad (97)$$

where k_{18}, k_{19}, k_{110}, r_{01}, k_{i8}, k_{i9}, k_{i10}, $r_{0i}(> 0)$ are design parameters. Then, we obtain the corresponding h_1, $h_i\,(\geq 2)$

$$\begin{aligned} h_1 &\leq \left\{\hat{b}_0^2 k_{18}\gamma_1^{*2} + (\hat{b}_0^2 k_{19}\gamma_1^{*2} - r_{10})\|\tilde{\omega}_1\|^2 \right.\\ &\quad \left. +(\hat{b}_0^2 k_{110}\gamma_1^{*2} - 4r_{10}\gamma_1^{*2})(\hat{\theta}_1 + \hat{\theta}_2\phi(z_1))\right\}/(4r_{10}\gamma_1^{*2}) \\ h_i &\leq \left[k_{i8}\gamma_i^{*2} + (k_{i9}\gamma_i^{*2} - r_{i0})\|\tilde{\omega}_i\|^2 \right.\\ &\quad \left. +(k_{i10}\gamma_i^{*2} - 4r_{i0}\gamma_i^{*2})\{k_{i1}\gamma_{i-1}^2 + k_{i2}\|\gamma_{Ki-1}\|^2\|\tilde{v}_1\|^2\}\right] \\ &\quad /(4r_{i0}\gamma_i^{*2}) \quad (i \geq 2) \end{aligned} \qquad (98)$$

In order that h_1, $h_i\,(\geq 2)$ are positive definite, k_{i9} and k_{i10} should be chosen such that

$$\begin{aligned} k_{19} &\geq \frac{r_{10}}{\hat{b}_0^2\gamma_1^{*2}} \;\Rightarrow\; k_{19} \geq \frac{r_{10}}{\delta^2\gamma_1^{*2}} \\ k_{110} &\geq \frac{4r_{10}}{\hat{b}_0^2} \;\Rightarrow\; k_{110} \geq \frac{4r_{10}}{\delta^2}, \quad (0 < \delta \leq b_0,\, \hat{b}_0) \\ k_{i9} &\geq \frac{r_{i0}}{\gamma_i^{*2}}, \quad k_{i10} \geq 4r_{i0} \quad (i \geq 2) \end{aligned} \qquad (99)$$

And, we get the explicit descriptions of the control inputs

$$\begin{aligned} v_1 &= -\frac{\hat{b}_0}{2r_1}z_1 \\ &= -\frac{\hat{b}_0\left[k_{18} + k_{19}\|\tilde{\omega}_1\|^2 + k_{110}\{\hat{\theta}_1 + \hat{\theta}_2\phi(z_1)\}\right]}{2r_{10}}z_1 \end{aligned} \qquad (100)$$

$$\begin{aligned} v_i &= -\frac{1}{2r_i}z_i \\ &= -\frac{k_{i8} + k_{i9}\|\tilde{\omega}_i\|^2 + k_{i10}\{k_{i1}\gamma_{i-1}^2 + k_{i2}\|\gamma_{Ki-1}\|^2\|\tilde{v}_1\|^2\}}{2r_{i0}}z_i \\ &\qquad (i \geq 2) \end{aligned} \qquad (101)$$

Solution II. Next, we obtain r_i and h_i by setting

$$h_1 = a_1 r_{11} + \frac{\hat{b}_0^2}{4r_{10}} \qquad (102)$$

$$h_i = a_i r_{i1} + \frac{1}{4r_{i0}} \quad (i \geq 2) \qquad (103)$$

$$\frac{1}{r_i} = \frac{1}{r_{i0}} + \frac{1}{r_{i1}} \qquad (104)$$

where a_i, $r_{i0}\,(0 < a_i, r_{i0} < \infty)$ are positive constants, which prescribe the ratios between r_i and h_i. Then, for equality condition of (34), (67), we obtain r_i as the positive solution of

$$a_1 r_{11}^2 + G_1 r_{11} - \frac{\hat{b}_0^2}{4} = 0 \qquad (105)$$

$$G_1 = \hat{\theta}_1 + \hat{\theta}_2\phi(z_1) + \frac{\|\tilde{\omega}_1\|^2}{4\gamma_1^{*2}} \qquad (106)$$

$$a_i r_{i1}^2 + G_i r_{i1} - \frac{1}{4} = 0 \quad (i \geq 2) \qquad (107)$$

$$G_i = k_{i1}\gamma_{i-1}^2 + k_{i2}\|\gamma_{Ki-1}\|^2\|\tilde{v}_1\|^2 + \frac{\|\tilde{\omega}_i\|^2}{4\gamma_i^{*2}} \quad (i \geq 2) \qquad (108)$$

Hence, r_{i1} and h_i are

$$\begin{aligned} r_{11} &= \frac{-G_1 + \sqrt{G_1^2 + a_1\hat{b}^2}}{2a_1} = \frac{\hat{b}_0^2}{2\left\{\sqrt{G_1^2 + a_1\hat{b}_0^2} + G_1\right\}} \\ r_{i1} &= \frac{-G_i + \sqrt{G_i^2 + a_i}}{2a_i} = \frac{1}{2\left\{\sqrt{G_i^2 + a_i} + G_i\right\}} \\ &\qquad (i \geq 2) \end{aligned} \qquad (109)$$

$$\begin{aligned} h_1 &= \frac{-G_1 + \sqrt{G_1^2 + a_1\hat{b}_0^2}}{2} + \frac{\hat{b}_0^2}{4r_{10}} \\ h_i &= \frac{-G_i + \sqrt{G_i^2 + a_i}}{2} + \frac{1}{4r_{i0}} \quad (i \geq 2) \end{aligned} \qquad (110)$$

and the explicit description of the control input is given by

$$v_1 = -\frac{\hat{b}_0}{2r_1}z_1 = -\left\{\frac{1}{\hat{b}_0}\left(\sqrt{G_1^2 + a_1\hat{b}_0} + G_1\right) + \frac{\hat{b}_0}{2r_{10}}\right\}z_1 \qquad (111)$$

$$v_i = -\frac{1}{2r_i}z_i = -\left\{\left(\sqrt{G_i^2 + a_i} + G_i\right) + \frac{1}{2r_{i0}}\right\}z_i \quad (i \geq 2) \qquad (112)$$

For those two explicit solutions of v_i, we have the following theorem.

Theorem 5. For those two solutions of v_i (**Solution I** and **Solution II**), the residual regions of z_i can be made arbitrarily small by proper choices of the design parameters k_{i8}, k_{i9}, k_{i10}, r_{i0}, a_i, γ_i^* (sufficiently large k_{i8}, k_{i9}, k_{i10}, a_i and sufficiently small r_{i0},, γ_i^*).

Proof. By the proper choices of the design parameters k_{i8}, k_{i9}, k_{i10}, r_{i0}, a_i, γ_i^*, the positive functions h_i (98), (110) can be made arbitrarily large, and $\gamma_1^{*2}\|\tilde{\Theta}_{1n^*}\|^2$, $\gamma_i^{*2}\|\tilde{\Theta}_2\|^2$ sufficiently small, while other terms remain unchanged in (92). Since it holds that

$$\begin{aligned} \frac{1}{T}\int_0^T \sum_{i=1}^{n^*} h_i z_i(t)^2 dt &\leq \frac{\sum_{i=1}^{n^*}\{V_i(0) - V_i(T)\} + N_0}{T} \\ &+ \frac{\gamma_1^{*2}\int_0^T \|\tilde{\Theta}_{1n^*}\|^2 dt + \sum_{i=2}^{n^*}\gamma_i^{*2}\int_0^T \|\tilde{\Theta}_2\|^2 dt}{T} \end{aligned} \qquad (113)$$

and h_i can be made arbitrarily large, then $\frac{1}{T}\int_0^T z_i(t)^2 dt$ can be made arbitrarily small.

4. Remarks

1. The nominal values Φ^*, θ_1^*, θ_2^* correspond to average values of time-varying Φ, θ_1, θ_2, respectively, and especially it holds that $\Phi^* = \Phi$, $\theta_1^* = \theta_1$, $\theta_2^* = \theta_2$ for time-invariant Φ, θ_1, θ_2. Furthermore, $\bar{b}_0$, $\underline{b}_0$ and $\bar{p}$, $\underline{p}$ determine the bounds in which time-varying b_0, p are included. Thus, $b_0 = \underline{b}_0 = \bar{b}_0$, $p = \underline{p} = \bar{p}$ for time-invariant b_0, p. Hence, when all system parameters are time-invariant, it follows that $\|\tilde{\Theta}_{1n^*}\| = \|\tilde{\Theta}_2\| = 0$, and that the exogenous disturbance terms in the $\mathcal{H}_\infty$ problem, are 0. On

the contrary, when several parameters are time-varying, $\|\tilde{\Theta}_{1n^*}\| \neq 0$ and/or $\|\|\tilde{\Theta}_2\| \neq 0$, and nonzero disturbance terms do exist.

2. Theorem 1 assures that $\mathcal{H}_\infty$ control schemes $v_1^* \sim v_{n^*}^*$ stabilize the overall control systems for arbitrary and bounded tuning parameters $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{p}$, $\hat{\Phi}$, $\hat{b}_0$, which are not necessarily determined by the proposed adaptive control schemes (13), (14), (15), (73), (74) in the manuscript (For example, the control systems are bounded even for fixed tuning parameters). For such case, D^* (the disturbance term) in (89) does not equal to 0, and the control errors do not converge to 0. On the contrary, when the proposed adaptive laws are utilized, the exogenous disturbance terms correspond to $\tilde{\Theta}_{1n^*}$, $\tilde{\Theta}_2$ which are time-varying elements included in system parameters (or $\|\tilde{\Theta}_{1n^*}\| = \|\tilde{\Theta}_2\| = 0$ for time-invariant system parameters). Hence, the proposed adaptive control schemes make the exogenous disturbance terms smaller than the non-adaptive (or the different adaptive) case. Additionally, it should be noted that Theorem 1 holds for arbitrary adaptation schemes, but Theorem 2 $\sim$ 5 hold for the proposed adaptation strategy in the manuscript. Asymptotic zero control errors for time-invariant case are attained by the proposed adaptive laws (13), (14), (15), (73), (74).

5. Numerical Example

In order to show the effectiveness of our proposed methods, the numerical simulation studies are performed. We consider adaptive tracking control problems for the simple process and control variable defined by

$$\begin{aligned} &\dot{x}(t) = \phi x_{f1}(t) + u_{f1}(t) \\ &\dot{x}_{f1}(t) = -\lambda x_{f1}(t) + x(t) \\ &\dot{u}_{f1}(t) = -\lambda u_{f1}(t) + u(t) \\ &(x(0) = x_{f1}(0) = u_{f1}(0) = 0) \\ &e(t) = x(t) - r(t) \\ &r(t) = \sin t \end{aligned}$$

where the relative degree $n^* = 2$, and ϕ can be time-invariant or time-varying. In the numerical simulations, bounded parameters ϕ (unknown) are chosen in the following:

$$\phi = \begin{cases} 10 & (\text{time} - \text{invariant}) \\ 5 + 5\sin 0.5t & (\text{time} - \text{varying}) \end{cases}$$

The design parameters are determined such that

$$\begin{aligned} &g = 100 \quad (\text{adaptive gains}) \\ &M = 100 \quad (\text{projection parameters}), \qquad \lambda = 1 \end{aligned}$$

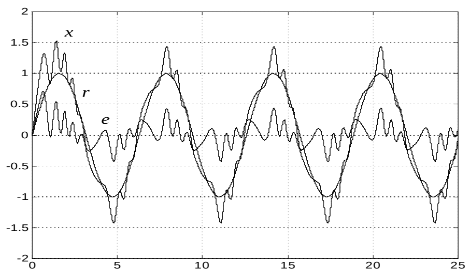

Fig. 1 Simulation result (time-invariant, conventional method)

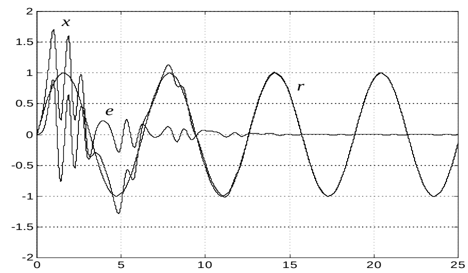

Fig. 2 Simulation result (time-invariant, proposed method I)

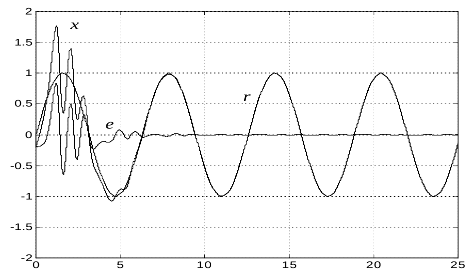

Fig. 3 Simulation result (time-invariant, proposed method II)

$$k_{ij} = 1, \quad r_{10} = r_{20} = 1, \quad a_1 = a_2 = 1, \quad \gamma_1^* = \gamma_2^* = 1$$

Fig.1, Fig.2 and Fig.3 show the time-invariant cases, while Fig.4, Fig.5 and Fig.6 show the time-varying case. For comparison, Fig.1 and Fig.4 show the results where conventional backstepping strategy (where the same k_{ij} are chosen) is adopted. It is seen that the better transient properties are given with less control efforts in the proposed control scheme.

6. Concluding Remarks

In the present paper, we proposed design methods of a new class of adaptive nonlinear $\mathcal{H}_\infty$ control systems for processes with bounded variations of parameters, by considering particular nonlinear $\mathcal{H}_\infty$ control problems, where unknown system parameters are regarded as exogenous disturbances to the processes. It is shown that the proposed control strategy can be applied to any time-varying

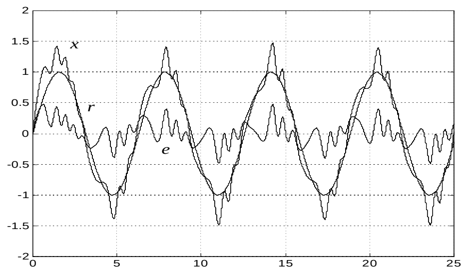

Fig. 4 Simulation result (time-varying, conventional method)

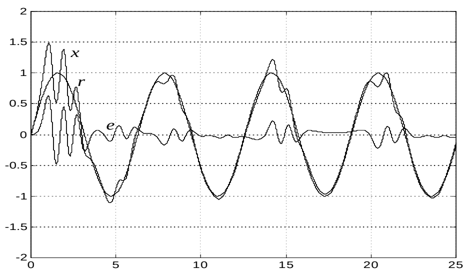

Fig. 5 Simulation result (time-varying, proposed method I)

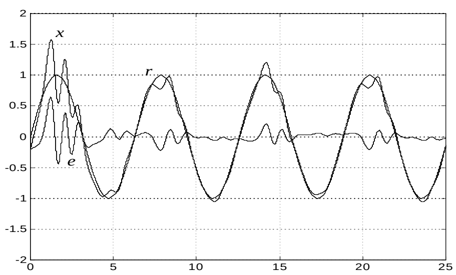

Fig. 6 Simulation result (time-varying, proposed method II)

(or time-invariant) systems, and the resulting control systems are bounded for arbitrarily large but bounded variations of time-varying parameters (Theorem 1). Also, the control schemes are shown to be sub-optimal to some $\mathcal{H}_\infty$ cost functionals (or certain differential games), when the high-frequency gains are time-invariant (Theorem 2), and even if that condition does not hold, the $\mathcal{L}_2$ gains from system parameters to generalized outputs are prescribed explicitly (Theorem 3). In the ideal case where time-varying parameters converge to time-invariant with the rate of $\mathcal{L}_2$ order, then the control variable converges to zero, that is, the zero residual control error is attained (Theorem 4). Finally, the explicit descriptions of the control schemes are given by assuming specified forms to the weighting functions h_i and r_i. The properties of those controllers are discussd (Theorem 5).

The merits of the proposed control strategy come form the interplay of $\mathcal{H}_\infty$ control and adaptive control, that is, the $\mathcal{H}_\infty$ control strategy stabilizes time-varying systems for arbitrary tuning parameters, and the adaptation scheme makes the exogenous disturbance terms smaller than the non-adaptive case.

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