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Robust Stabilization of Multivariable High Gain Feedback Systems†

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In this paper we consider a design problem of multivariable high gain feedback systems with robust stability.

High gain feedback control has many advantages for system performances, while it was reported that plant perturbations often cause instability of high gain feedback systems. Hence we discuss a robust stabilization problem of high gain feedback systems.

The plant is assumed to belong to the multiplicative output perturbation class $\mathbf{M}(P_0, r) := \{P : P = (I + L)P_0,\ ||L(j\omega)|| \leq |r(j\omega)|,\ \forall\omega$, where L doesn't change the number of unstable poles $\}$. Here P_0 denotes the $m \times p$ nominal plant and r denotes the bound of perturbations. The system contains high gains (not necessarily linear gains) in each feedback loop.

We say a system to be roubst positive real if $PC(I + PC)^{-1}$ remains stable and positive real for all plant P belonging to $\mathbf{M}(P_0,\ r)$, where C denotes a compensator. Obviously the robust positive realness (RPR) guarantees the stability of the system for all P belonging to $\mathbf{M}(P_0,\ r)$ and for all nonlinear gains (while D.C. gains ≥ 1).

We assume rank $P_0 = m$, $\psi(N_0)$ and $\psi(D_0)$ are coprime, where $P_0 = N_0 D_0^{-1}$ is a right coprime factorization over the ring of stable real rational matrices and $\psi(Q)$ denotes the largest invariant factor of Q. Moreover, we assume the roots of $|r(j\omega)|^2 - 1 = 0$ are finitely many and their multiplicities ≤ 2.

Under these conditions there exists a proper compensator which attains the RPR and rank $P_0 C = m$ if and only if the following two conditions are satisfied:

1) $\psi(N_0)$ has no finite zeros in the open right-half plane and has no multiple $j\omega$-axis zeros (including $j\infty$), and

2)

$$|r(j\omega)| = \begin{cases} = 0, & \text{for } j\omega\text{-axis zeros of } \psi(N_0), \\ < 1, & \text{for } j\omega\text{-axis zeros of } \psi(D_0), \\ \leq 1, & \text{elsewhere.} \end{cases}$$

A numerical example is given in order to show that RPR control copes with the perturbations which cause instability for an LQ optimal control system.

Key Words: robust stability, multivariable system, high gain system, robust positive realness

1. Introduction

High gain feedback control seems to have advantages of good disturbance rejection, good tracking performance, and easy realization of decoupling of the input/output variables for a multivariable system. Regarding the high gain feedback control, infinite gain margin problems have been discussed in the literatures [1)~3)] where the realization conditions for the infinite gain margin and the design procedures of the compensators have been derived.

In general, a plant model includes uncertainties originated from identification errors, the model simplification for the purpose of compensator design, the variation of the plant parameters due to the long time operation and/or environment changes. These uncertainties can be treated as perturbations of the nominal plant.

It has been reported that high gain feedback control is easy to make the system unstable against even small variation of the parameters through an example of cheap optimal LQ control [4)]. In order to make high gain feedback control more reliable, in this paper we consider a design problem of multivariable high gain feedback systems with robust stability for the plant perturbations.

A robust stabilization problem for single input/single output high gain feedback control has been already discussed in the literature 5). This paper extends the results to the case of multivariable systems. As in the single input/single output case, the robust stability for the high gain feedback system is treated in terms of the robust pos-

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itive realness(RPR) [5]. An existence condition of a compensator to attain the RPR is derived, and an allowable range of the plant perturbation is described in detail.

It is also shown that a realization of the RPR attains at the same time what we call a robust performance, that is, the sensitivity function of the feedback system is suppressed less than or equal to 1 over all the frequency [6]. Through an example it is demonstrated that a compensator designed in this paper's manner can stabilize the plant perturbations which cause instability for the conventional LQ optimal control.

2. Robust Positive Realness

Through this paper, let $\mathrm{R}(s)$ denote the set of all proper real rational functions of s, and $\mathcal{S}$ a subset of $\mathrm{R}(s)$ whose element has no poles in the extended right-half plane. The sets of matrices whose elements belong to $\mathrm{R}(s)$ and $\mathcal{S}$ are represented by $\mathrm{M}(\mathrm{R}(s))$ and $\mathrm{M}(\mathcal{S})$, respectively.

Now, let us consider the feedback system in Fig. 1, where $P(s)$ and $C(s) \in \mathrm{M}(\mathrm{R}(s))$ denote the transfer matrix of the $m \times p$ plant, and that of the $p \times m$ compensator. It is assumed that each feedback path has a nonlinear and time invariant gain factor. In Fig. 1, $K(\cdot) = \mathrm{diag}\ \{k_i(\cdot)\}$ represents the gain matrix where the condition of (1) is satisfied:

$$\infty > k_i(y_i)/y_i \geq 1, \quad i = 1, \ldots, m. \tag{1}$$

As well known, if C stabilizes P and $A := PC(I + PC)^{-1} \in \mathrm{M}(\mathcal{P})$, then the feedback system in Fig. 1 is stable for any gain matrix $K(\cdot)$ satisfying (1) [7], where $\mathrm{M}(\mathcal{P})$ denotes the set of all positive real matrices. A necessary and sufficient condition for a matrix to be positive real is given as follows.

Lemma 1. [8] A square matrix $A(s) \in \mathrm{M}(\mathrm{R}(s))$ is a positive real matrix if and only if the following four conditions hold:

i) $A(s)$ has no poles in the open right-half plane $\{s : \mathrm{Re}\ s > 0\}$,

ii) $A(s)$ has no multiple poles on the $j\omega$-axis,

iii) all residual matrices of A at the $j\omega$-poles are Hermitian positive semi-definite, and

iv) $A_H(j\omega)$ is positive semi-definite for all ω except for the poles,

where $A_H(s) := (A^*(s) + A(s))/2$, and $A^*(s)$ denotes the complex conjugate transpose matrix of $A(s)$.

In this paper, firstly, the plant uncertainty is assumed to be described in the form of the multiplicative perturbation at the plant output. Here let us define the multiplicative

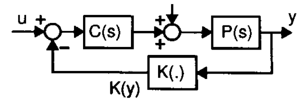

Fig. 1 Feedback System

output perturbation class $\mathbf{M}(P_0,\ r)$:

$$\begin{aligned}\mathbf{M}(P_0, r) := \{P : P = (I + L)P_0,\ &||L(j\omega)|| \leq |r(j\omega)|,\\ &\forall\omega \in \mathrm{R},\ L(s) \in \mathrm{M}(\mathrm{R}(s)),\\ &P(s) \text{ has the same number of the}\\ &\text{unstable poles as that of } P_0(s)\},\end{aligned}$$

where $P_0 \in \mathrm{M}(\mathrm{R}(s))$ denotes the nominal model of the plant dynamics, and $r(s) \in \mathcal{S}$ characterizes the allowable range of the multiplicative perturbation. Also $||A||$ denotes the largest singular value of a matrix A.

Now let us define the robust positive realness.

Definition 1. The system class $(P_0,\ C,\ r)$ in Fig. 1 is robustly positive real or has the robust positive realness if for all $P \in \mathbf{M}(P_0,\ r)$

C stabilizes P, and

$A := PC(I + PC)^{-1} \in \mathrm{M}(\mathcal{P})$,

where the system class $(P_0,\ C,\ r)$ is defined by the plant class $\mathbf{M}(P_0,\ r)$ and the fixed compensator C.

3. Robust Positive Real Control

3.1 Preparation

The following Lemma 2 and 3 are both well known; Lemma 2 describes a robust stabilization condition, and Lemma 3 does the relationship between a positive real matrix and a bounded real one, respectively.

Lemma 2. [9] C stabilizes all the plant $P \in \mathbf{M}(P_0,\ r)$ if and only if

C stabilizes P_0, and

$|r(j\omega)| \cdot ||A_0(j\omega)|| < 1,\ \forall\omega \in \mathrm{R}_e := \mathrm{R} \cup \{\infty\}$,

where $A_0 := P_0C(I + P_0C)^{-1}$.

Lemma 3. [8] A square matrix $A \in \mathrm{M}(\mathrm{R}(s))$ is a positive real matrix if and only if

there exists $\Lambda = (I - A)(I + A)^{-1}$, and

Λ is a bounded real matrix, i.e.,

$\Lambda \in \mathrm{M}(\mathcal{S})$, and

$||\Lambda(j\omega)|| \leq 1,\ \forall\omega \in \mathrm{R}$.

For the purpose of easy handling of the problem, let us introduce coprime factorization forms [10] of $P_0 \in \mathrm{M}(\mathrm{R}(s))$ over $\mathrm{M}(\mathcal{S})$ as

$$P_0 = N_0D_0^{-1} = \tilde{D}_0^{-1}\tilde{N}_0,$$

where

$$X_0N_0 + Y_0D_0 = I, \quad \tilde{N}_0\tilde{X}_0 + \tilde{D}_0\tilde{Y}_0 = I,$$

$N_0, D_0, \tilde{N}_0, \tilde{D}_0, X_0, Y_0, \tilde{X}_0, \tilde{Y}_0 \in \mathrm{M}(\mathcal{S})$.

Then let us define a set

$$\mathcal{A}(P_0) := \{A_0 : A_0 = N_0(X_0 + R\tilde{D}_0), R \in \mathrm{M}(\mathcal{S})\}.$$

When $A_0 \in \mathrm{M}(\mathcal{S})$ belongs to $\mathcal{A}(P_0)$, A_0 is said to be feedback realizable[11]. The following Lemma 4 is useful regarding the feedback realizability.

Lemma 4.[2] If $A_0 \in \mathcal{A}(P_0)$ and $\det A_0 \neq 0$ then $\psi(N_0)|\psi(A_0)$ and $\psi(\tilde{D}_0)|\psi(I - A_0)$. Conversely, if $a_0 \in \mathcal{S}$ satisfies that $\psi(N_0)|a_0$ and $\psi(\tilde{D}_0)|(1 - a_0)$ then $A_0 := a_0 I \in \mathcal{A}(P_0)$, where $\psi(Q)$ denotes the largest invariant factor of $Q \in \mathrm{M}(\mathcal{S})$[10].

For a given A_0, if P_0 has the full column rank, a compensator is given by

$$C = D_0 N_0^{-R} A_0 (I - A_0)^{-1} \quad (2)$$

where N_0^{-R} denotes the right inverse matrix of N_0 over the real rational matrix ring. Therefore, once $A_0 \in \mathcal{A}(P_0)$ is suitably specified, design of C is completed by (2).

3.2 Realizability of RPR and Compensator Design

A main interest of this paper is to derive a realizability condition of the RPR control for the multivariable system. This is stated in Theorem 1 under Assumption 1.

Assumption 1. Let us assume that rank $P_0 = m$, and that $\psi(N_0)$ and $\psi(Q_0)$ are coprime. Additionally, the roots of $|r(j\omega)|^2 - 1 = 0$ are finitely many and their multiplicities are less than or equal to 2.

Theorem 1. Under Assumption 1, there exists a proper compensator $C \in \mathrm{M}(\mathrm{R}(s))$ which realizes the RPR and rank $P_0 C = m$ if and only if the following conditions hold:

$$\text{(i) } \psi(N_0) \in \mathcal{M}_1 \text{ and} \quad (3a)$$

$$\text{(ii) } |r(j\omega)| \begin{cases} = 0, & \omega \in \Omega_n \quad (3b) \\ < 1, & \omega \in \Omega_d \quad (3c) \\ \leq 1, & \text{other } \omega \in \mathrm{R} \quad (3d) \end{cases}$$

where $\mathcal{M}_1, \Omega_n$ and Ω_d are defined as

$$\mathcal{M}_1 := \{f \in \mathcal{S} : f \text{ has no zeros in the open right half plane, and } f \text{ has no multiple zeros on the } j\omega\text{-axis including } j\infty\},$$

$$\Omega_n := \{\omega \in \mathrm{R}_e := \mathrm{R} \cup \{\infty\} : \psi(N_0)(j\omega) = 0\},$$

$$\Omega_d := \{\omega \in \mathrm{R} : \psi(\tilde{D}_0)(j\omega) = 0\}.$$

According to Theorem 1, in order to realize the RPR, the nominal plant and the plant uncertainty range are restricted as follows:

- No zeros of the nominal plant are allowed in the right-half plane; no multiple zeros on the $j\omega$-axis(including $j\infty$) are allowed in terms of the largest invariant factor.
- The plant model uncertainties at the $j\omega$-zeros are not allowed, if $j\omega$-zeros of the nominal plant exist.
- The plant model uncertainties at the $j\omega$-poles must be less than 100% to the nominal model, if $j\omega$-poles of the nominal plant exist.
- The plant model uncertainties must be less than or equal to 100% to the nominal model on the $j\omega$-axis except for the $j\omega$-zeros and $j\omega$- poles of the nominal plant.

Proof

It is useful to note that by Definition 1, Lemma 2 and 3, the system class (P_0, C, r) is robust positive real if and only if

$$C \text{ stabilizes } P_0, \text{ and} \quad (4a)$$

$$|r(j\omega)| \cdot ||A_0(j\omega)|| < 1, \forall \omega \in \mathrm{R}_e, \quad (4b)$$

and for all $L \in \mathcal{L}(r)$ there exists

$$\Lambda = (I - A)(I + A)^{-1}$$
$$(I - T_0)(I + LT_0)^{-1}$$

such that

$$\Lambda \in \mathrm{M}(\mathcal{S}) \text{ and} \quad (4c)$$

$$||\Lambda(j\omega)| \leq 1, \forall \omega \in \mathrm{R}, \quad (4d)$$

where

$$\mathcal{L}(r) := \{L \in \mathrm{M}(\mathrm{R}(s)) : ||L(j\omega)|| \leq |r(j\omega)|, \forall \omega \in \mathrm{R}, \text{ the number of the plant unstable poles is same as that of } P_0\} \text{ and}$$

$$T_0 := 2P_0 C(I + 2P_0 C)^{-1}.$$

Let us give the proof of Theorem 1 based on the above fact.

"Sufficiency" It is enough to show that there exists a proper compensator C satisfying all the conditions of (4) under the assumptions of (3) and Assumption 1. This will be completed by firstly designing a compensator C_0 which attains infinite gain margin (Step 1)[2], and secondly specifying the gain factor $\kappa(\geq 1)$ for the desired compensator $C := \kappa C_0$ so as to attain the robust stability (Step 2 ∼ Step 6).

Step 1 The first step is to find a solution for the following interpolation problem. As shown in Lemma B in Appendix of the literature 5), this problem has a solution.

[**Interpolation Problem**] Find a stable positive real function $\eta(s)$ satisfying the following six conditions:

(I) $\eta(j\omega_{n_i}) = 0,\ \omega_{n_i} \in \Omega_n$,

(II) $\eta(q_i) = 1,\ q_i \in Q$,

(III) $\eta^{(k)}(q_i) = 0,\ k = 1, \ldots, m_i - 1,\ q_i \in Q$,

(IV) $\eta(j\omega_{r_i}) =: \beta_i,\ 1 > \beta_i > 0,\ \omega_{r_i} \in \Omega_r$,

(V) $\eta(j\omega) \neq 0,\ \forall \omega \in \mathrm{R} \setminus \Omega_n$, and

(VI) $\eta(j\infty) \neq 1$,

where

$$Q := \{\mathrm{Re}\, q_i \geq 0 : \psi(\tilde{D}_0)(q_i) = 0,\ q_i \text{ is of } m\text{-th order}\},$$

$$\Omega_r := \{\omega \in \mathrm{R}_e : |r(j\omega)| = 1\}.$$

From Lemma 4, the interpolation conditions (I) $\sim$ (III) imply $\eta I \in \mathcal{A}(P_0)$, and this η gives

$$C_0 := D_0 N_0^{-R} \eta/(1-\eta) \quad (5)$$

where it is noted that N_0^{-R} exists since P_0 has the row full rank.

It is easy to see that, for the interpolation conditions (I) and (VI), C_0 given by (5) is proper. Now let the desired compensator be in the form of

$$C := \kappa C_0 \quad (6)$$

and find a suitable $\kappa(\geq 1)$ by the following procedure.

Step 2 Let us find $\kappa(\geq 1)$ satisfying

$$|r(j\omega)| \cdot ||A_0(j\omega)|| + ||I - A_0(j\omega)|| \leq 1, \forall\omega \in \mathrm{R}. \quad (7)$$

Since

$$A_0 = P_0 C(I + P_0 C)^{-1} = \kappa\eta/\{(\kappa-1)\eta + 1\} I =: a_0 I,$$

clearly (7) is equivalent to the following inequality:

$$|r(j\omega)| \cdot |a_0(j\omega)| + |1 - a_0(j\omega)| \leq 1, \forall\omega \in \mathrm{R}, \quad (8)$$

and so as to (4.4) in the literature 5). As shown in the literature [5)], κ satisfying (8) can be found. Let κ_0 denote this value of κ, then the next several steps describe $\kappa_0 C$ is desired one.

Step 3 Since ηI is a stable positive real matrix, C_0 given by (5) stabilizes the feedback system consisting of P_0 and C for the gain of (1). Therefore, for $\kappa \geq 1$, κC_0 is a stabilization compensator for the nominal plant P_0, and this implies (4a).

Step 4 This step shows that (4b) is satisfied. Two cases are considered.

(1) For ω such that $\eta(j\omega) \neq 1$, it is shown that

$$||I - A_0(j\omega)|| = \left| \frac{1 - \eta(j\omega)}{(\kappa_0 - 1)\eta(j\omega) + 1} \right| > 0.$$

Thus, by using (7), (4b) is obvious.

(2) For ω such that $\eta(j\omega) = 1$, it is clear that $||A_0(j\omega)|| = |\eta(j\omega)| = 1$. Then by using the interpolation condition (IV) and (3d), it is derived that $|r(j\omega)| < 1$ for such ωs. Hence (4b) holds.

Step 5 This step shows (4d) is satisfied. Since (8) holds for all $\kappa \geq \kappa_0$, it obviously holds for a specific case of $\kappa = 2\kappa_0$. By noting that for $\kappa = \kappa_0$

$$T_0 = 2P_0 C(I + 2P_0 C)^{-1} = 2\kappa_0\eta/\{(2\kappa_0 - 1)\eta + 1\} I,$$

it is easy to see that

$$|r(j\omega)| \cdot ||T_0(j\omega)|| + ||I - T_0(j\omega)|| \leq 1, \forall\omega \in \mathrm{R}. \quad (9)$$

Moreover,

$$|r(j\omega)| \cdot ||T_0(j\omega)|| < 1, \forall\omega \in \mathrm{R}_e \quad (10)$$

is derived through a similar manner as in Step 4. From (9) and (10), for all $L \in \mathcal{L}(r)$

$$||\Lambda(j\omega)|| = ||(I - T_0(j\omega))(I + LT_0(j\omega))^{-1}|| \leq 1, \forall\omega \in \mathrm{R}.$$

This implies (4d).

Step 6 The last step shows that (4c) holds for all $L \in \mathcal{L}(r)$. From (9) it is easy to see

$$||I - T_0(j\omega)|| = ||(I - A_0)(I + A_0)^{-1}(j\omega)|| \leq 1, \forall\omega \in \mathrm{R}. \quad (11)$$

By noting that $A_0 = a_0 I$, (11) implies $\mathrm{Re} a_0(j\omega) \geq 0, \forall\omega \in \mathrm{R}$. Additionally, since $a_0 \in \mathcal{S}$, it follows that $a_0 I$ is a stable positive real matrix. According to the passive theorem, the feedback system is stable for the linear gain $k_i(y_i)/y_i = 2$, $i = 1, \ldots, m$, and therefore $2C$ is a stabilization compensator for the nominal plant P_0. Based on this statement and (10), $2C$ is a robust stabilizer for a class of $\mathbf{M}(P_0, r)$, and $\Lambda = (I + 2(I + L)P_0 C)^{-1} \in \mathrm{M}(\mathcal{S})$ for all $L \in \mathcal{L}(r)$.

"Necessity" The necessity part of proof is given in Appendix. ∎

3.3 Robust Performance of RPR control

The compensator designed above realizes additionally the robust performance, that is, the sensitivity function $||S(j\omega)|| := ||((I + PC)(j\omega))^{-1}||$ is less than or equal to 1 for all the frequency regardless of the multiplicative perturbation at the plant output, even when the channel gain is nominal, i.e., $k_i(y_i)/y_i = 1$, $i = 1, \ldots, m$, as described below.

Actually, it is easy to derive that for all $P \in \mathbf{M}(P_0, r)$ and for all $\omega \in \mathrm{R}$,

$$\begin{aligned} ||S(j\omega)|| &= ||(I - A_0(j\omega))(I + (LA_0)(j\omega))^{-1}|| \\ &\leq ||I - A_0(j\omega)||/\{1 - |r(j\omega)| \cdot ||A_0(j\omega)||\} \\ &\leq 1, \end{aligned}$$

where (7) is applied to derive the third line.

Since the robust stability is guaranteed for higher gains, the sensitivity performance can be improved by having higher gains, depending on demands.

3.4 Multiplicative Input Perturbation

Here, let us consider the feedback system depicted in Fig. 2, where the plant with perturbation is assumed to

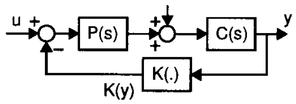

Fig. 2 Feedback System

belong to the class $\mathbf{N}(r,\ P_0)$, instead of $\mathbf{M}(P_0, r)$, representing the multiplicative uncertainty at the plant input:

$$\begin{aligned}\mathbf{N}(r,\ P_0) := \{&P : P = P_0(I+L), ||L(j\omega)|| \leq |r(j\omega)|,\\ &\forall\omega \in \mathrm{R}, L(s) \in \mathrm{M}(\mathrm{R}(s)),\\ &P \text{ has the same number of the unstable}\\ &\text{poles as that of } P_0\}.\end{aligned}$$

As in (1) the nonlinear gain elements of $K(\cdot) := \mathrm{diag}\{k_i(\cdot)\}$ satisfy

$$\infty > k_i(y_i)/y_i \geq 1,\ i = 1, \ldots, p. \tag{12}$$

Definition 2. The system class $(r,\ P_0,\ C)$ in Fig. 2 is robustly positive real or has the robust positive realness (for the multiplicative input perturbation), if for all $P \in \mathbf{N}(r,\ P_0)$

C stabilizes P, and

$A := CP(I+CP)^{-1} \in \mathrm{M}(\mathcal{P})$,

where the system class $(r,\ P_0,\ C)$ is defined by the plant class $\mathbf{N}(r,\ P_0)$ and the fixed compensator C.

This robust positive realness guarantees that the feedback system in Fig. 2 is stable for all plants belonging to $\mathbf{N}(r,\ P_0)$ and all nonlinear gains $K(\cdot)$ of (12).

Here Assumption 2 is put for Assumption 1 .

Assumption 2. Assume that rank $P_0 = p$, and that $\psi(\tilde{N}_0)$ and $\psi(\tilde{D}_0)$ are coprime. Also assume that the roots of $|r(j\omega)|^2 - 1 = 0$ are finitely many, and their multiplicities are at most 2.

Then Corollary 1 holds through a similar discussion as in Theorem 1.

Corollary 1. For the feedback system in Fig. 2, Theorem 1 holds, where the term of rank $P_0C = m$ is replaced by rank $CP_0 = p$.

The compensator is given by

$$C(s) = \kappa_0\eta/(1-\eta)\tilde{N}_0^{-L}\tilde{D}_0,$$

where $\eta(s)$ is the solution of **Interpolation Problem**, the compensator gain κ_0 is given by Step 2 in the proof of sufficiency for Theorem 1, and $\tilde{N}_0^{-L}$ denotes the left inverse matrix of $\tilde{N}_0$.

Additionally, the feedback system shows the robust performance against the multiplicative perturbations at the plant input, as described in **3.3**.

4. Examples

[Example 1] For a given $\mathbf{M}(P_0,\ r)$, let us design a compensator C to realize the robust positive realness.

Assume that

$$P_0 = \begin{bmatrix} \frac{s^3+3s^2+2s+1}{(s+1)^3(s+2)} & \frac{s}{(s+1)^3} \\ \frac{-s}{(s+1)^2(s+2)} & \frac{s}{(s+1)^2} \end{bmatrix}$$

and

$$r = \frac{s}{s^2+s+1}.$$

The right coprime factorization of P_0 is given by

$$N_0 = \begin{bmatrix} 1/(s+2) & s/(s+1)^3 \\ 0 & s/(s+1)^2 \end{bmatrix} \text{ and}$$

$$D_0 = \begin{bmatrix} 1 & 0 \\ 1/(s+2) & 1 \end{bmatrix}.$$

Since $\psi(N_0) = s/(s+1)^2 \in \mathcal{M}_1$, $|r(j\omega)| \leq 1$, $\forall\omega \in \mathrm{R}$, and $r(j\omega) = 0$ at $j\omega = j0$ and $j\infty$ which are the zeros of $\psi(N_0)$, the desired compensator can be designed by Theorem 1. Solving the interpolation problem, we have a solution as

$$\eta(s) = s/2(s^2+s+1),$$

and calculating κ_0, we obtain $\kappa_0 = 3$. Hence C is designed as

$$\begin{aligned} C &= 3\eta(1-\eta)D_0N_0^{-1} \\ &= \begin{bmatrix} \frac{3s(s+2)}{2s^2+s+2} & \frac{-3s(s+2)}{(2s^2+s+2)(s+1)} \\ \frac{3s}{2s^2+s+2} & \frac{3(s^3+3s^2+2s+1}{(2s^2+s+2)(s+1)} \end{bmatrix}. \end{aligned}$$

[Example 2] In this example, the RPR control is applied to a state-space feedback system. In Fig. 2, let us assume

$$P_0 = (sI-A)^{-1}B,$$

where $(A,\ B)$ is a stabilizable pair. Then

$$\begin{aligned}\tilde{D}_0 &= (sI-A)/(s+1) \text{ and}\\ \tilde{N}_0 &= B/(s+1)\end{aligned}$$

is a left coprime factorization over $\mathrm{M}(\mathcal{S})$. Obviously $\psi(\tilde{N}_0) \in \mathcal{M}_1$ and this guarantees the existence of a compensator C to realize the infinite gain margin for the nominal plant [3)] (According to the LQ control theory, it is well known that C can be a constant so as to attain the infinite gain margin). Additionally, Corollary 1 implies that there exists a robust stabilization compensator which realizes the infinite gain margin not only for the nominal plant but also for the multiplicative plant perturbations at the plant input within the range of (3). In this case, C is required to be a dynamical compensator as demonstrated in Example 3.

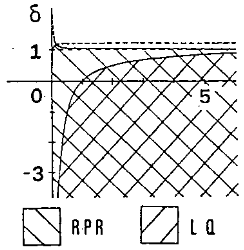

Fig. 3 Stable region

[Example 3] On a regulator problem, let us compare the RPR control with the conventional LQ optimal control from the stability point of view against parameter perturbations. Here assume in the feedback system in Fig. 2

$$p_0 = 1/(s-1) \text{ and} \quad (13)$$
$$r = 1/(s+1).$$

It is easy to see the conditions of (3) are all satisfied and hence the RPR control is available. The design procedure gives the compensator as

$$c(s) = 63(5s+1)/(2s-1).$$

With the above compensator, the nonlinear infinite gain margin is attained for the perturbed plant of

$$p = p_0\{1 - \delta/(s+1)\} \quad (14)$$

if $-1 \le \delta \le 1$. Especially, when k is additionally assumed to be linear, the characteristic equation of the closed loop system in Fig. 2 is given by

$$\Delta = [2s^3 + (315k-1)s^2 + \{63k(6-5\delta) - 2\}s + 63k(1-\delta) + 1]/(s+1)^3.$$

The stable region in the δ - k diagram is illustrated by the portion shaded with oblique lines from the upper left to lower right in Fig. 3.

On the other hand, in order to design a compensator for the LQ optimal control, let the system (13) be represented by a state-space realization of

$$A = 1, b = 1, c = 1.$$

The optimal gain $c(s) = k$ is obtained by solving the algebraic Riccati equation under suitable weight coefficients. For the perturbed plant of (14), the characteristic equation is

$$\Delta = \{s^2 + ks + k(1-\delta) - 1\}/(s+1)^2.$$

The stable region for δ and k is shown by the portion shaded with oblique lines from the upper right to lower left in Fig. 3. It is well known that the optimal regulator attains the infinite gain margin for the nominal plant (corresponding to the case of $\delta = 0$). However, according to the results in Fig. 3, when δ becomes closer to 1, the allowable range of k becomes narrower. Especially, in the case of $\delta = 1$, the feedback system cannot avoid instability regardless of the value of k, in other words, regardless of the weight coefficients for the Riccati equation.

Through the example, it is observed that the RPR control realizes a wider stable region for the δ - k variations compared to the LQ optimal control as shown in Fig. 3.

5. Conclusion

This paper has discussed the robust stabilization problem for multivariable high gain feedback systems, where the problem has been reduced to the realization problem of the robust positive realness of the system. The existence condition of a compensator attaining the robust positive realness is derived in terms of the conditions on the largest invariant factor for the numerator in the coprime factorization form of the nominal plant model $\psi(N_0)$, and in terms of those on the allowable range of the model uncertainty or perturbation $|r(j\omega)|$.

The results show that the infinite gain margin considerably gives constraints on the allowable range of the plant perturbation, though this might be slightly surprising since in a conventional understanding a feedback system with a large gain margin seems to attain better robust stability against the plant perturbation. This can be regarded as a trade-off between the robustness against the gain perturbation and that against the unstructured plant perturbation.

Also it has been shown in this paper that the RPR control makes the sensitivity function of the feedback system less than or equal to 1 over all the frequency.

Moreover, through the simple examples the RPR control has been compared to the conventional LQ control, and it has been demonstrated that the RPR control of a dynamical compensation scheme is more robustly stable against the unstructured plant uncertainty than the conventional LQ optimal control of a statical one.

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Appendix A. Proof of Necessity Part for Theorem 1

Step 1 Derivation of (3a). For rank $P_0C = m$, there exists A_0^{-1}, and it is also a positive real matrix [8]. Now let A_0 be expressed as

$$A_0^{-1} = \hat{A}_0/\psi(A_0),$$

where $\hat{A}_0 \in \mathrm{M}(\mathcal{S})$ and $\psi(A_0) \in \mathcal{S}$ are coprime. From Lemma 1, it follows that $\psi(A_0) \in \mathcal{M}_1$. Additionally, by using Lemma 4 on the feedback realizability, $\psi(N_0) \in \mathcal{M}_1$ is derived.

Step 2 Derivation of (3b). Under the assumption of $|r(j\omega_n)| > 0,\ \omega_n \in \Omega_n$, it can be shown that there exists $L \in \mathcal{L}(r)$ which does not satisfy Lemma 1. To do this, firstly, let $A_0^{-1}(j\omega)$ be approximated at the neighborhood of $j\omega_n$ in the form of

$$A_0^{-1}(j\omega) \simeq Z/\{j(\omega - \omega_n)\},\ (Z \neq 0).$$

From Lemma 1, Z is positive semi-definite, and hence there exists i such that $z_{ii} > 0$.

Now let us define

$$m(s) := \varepsilon r(s) \prod_i (s - \alpha_i)/(s + \alpha_i)$$

and $\alpha_i > 0$ such that $m(j\omega_n) = -\varepsilon|r(j\omega_n)|$. Then by using this $m(s)$, let us define $L(s)$ as

$$L(s) := e_i e_i' m(s),$$

where e_i is an unit vector whose i-th element is 1, and make ε sufficiently small so that $(I + L)N_0$ and D_0 are coprime. Then obviously $L \in \mathcal{L}(r)$, and $(I + L(j\omega_0))^{-1}$ exists since $||L(j\omega_0)|| < 1$ holds where ω_0 is the neighborhood of $\omega_n(\omega_0 \neq \omega_n)$. Here calculate a quadratic form $e_i' A^{-1}(j\omega) e_i$, then its real part is negative:

$$\mathrm{Re}\, e_i' A^{-1}(j\omega) e_i \simeq \frac{-\varepsilon|r(j\omega_n)|}{1 - \varepsilon|r(j\omega_n)|} < 0.$$

This contradicts the condition (iv) in Lemma 1.

Step 3 Derivation of (3c). Since $\omega_d \in \Omega_d$ is a pole of P_0, there exists $\xi(\neq 0) \in C^m$ to satisfy $(I - A_0(j\omega_d))\xi = 0$. Therefore, it follows that $||A_0(j\omega_d)|| \geq 1$. By noting that $|r(j\omega_d)| \cdot ||A_0(j\omega_d)|| < 1$ from (4b), it is clear to obtain (3c).

Step 4 Derivation of (3d). As in Step 2, let us use reduction to absurdity. Let us assume $|r(j\omega_0)| > 1$ for some $\omega_0 \in \mathrm{R} \setminus (\Omega_n \cup \Omega_d)$, it is shown that there exists $L \in \mathcal{L}(r)$ contradicting (4d).

With unitary matrices U and V, a singular value decomposition of $T_0(j\omega)$ is given by

$$UT_0(j\omega)V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Sigma := \mathrm{diag}\{\sigma_i\},\ i = 1, \ldots, n$, and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ are the singular values of $T_0(j\omega_0)$. Without loss of generality, we can assume $\sigma_1 > 0$. This is because if $\sigma_1 = 0$, i.e., $T_0(j\omega_0) = 0$, then we can alternatively choose $j\omega_0'$ in the neighborhood of $j\omega_0$ so that $T_0(j\omega_0') \neq 0$, and thus we can assume $T_0(j\omega_0') \neq 0$ and $|r(j\omega_0')| > 1$.

Let $r(j\omega_0)$ be expressed as $|r(j\omega_0)|\exp(j\delta)$, and let u and v denote the first row of U and the first column of V, respectively. Each element of u and v can be written by

$$u_i = \mu_i \exp(j\theta_i) \text{ and}$$
$$v_i = \nu_i \exp(j\phi_i),$$

where μ_i and ν_i are real. Then let us define

$$L := -r(s)\tilde{v}(s)\tilde{u}'(s), \tag{A.1}$$

where

$$\tilde{v}(s) := \begin{bmatrix} \nu_1 \prod_i (s - \alpha_{1i})/(s + \alpha_{1i}) \\ \vdots \\ \nu_m \prod_i (s - \alpha_{mi})/(s + \alpha_{mi}) \end{bmatrix},$$

$$\tilde{u}(s) := \begin{bmatrix} \mu_1 \prod_i (s - \beta_{1i})/(s + \beta_{1i}) \\ \vdots \\ \mu_m \prod_i (s - \beta_{mi})/(s + \beta_{mi}) \end{bmatrix}$$

and $\alpha_{ji},\ \beta_{ji} > 0$ can be decided to satisfy

$$arg\{\prod_i (j\omega_0 - \alpha_{ji})/(j\omega_0 + \alpha_{ji})\} = \phi_j - \delta/2$$

$$arg\{\prod_i (j\omega_0 - \beta_{ji})/(j\omega_0 + \beta_{ji})\} = \theta_j - \delta/2.$$

According to the above manner, L belongs to $\mathrm{M}(\mathcal{S})$, and $||L(j\omega)|| \leq |r(j\omega)|,\ \forall\omega \in \mathrm{R}$, and hence $L \in \mathcal{L}(r)$ follows. However, this L does not satisfy (4d) as shown in the follows.

$$||(I - T_0(j\omega_0))(I + L(j\omega_0)T_0(\omega_0))^{-1}||$$

$$
\begin{aligned}
&\geq \Big| ||I+L(j\omega_0)T_0(j\omega_0))^{-1}|| \\
&\quad - ||T_0(j\omega_0)(I+L(j\omega_0)T_0(j\omega))^{-1}|| \Big| \\
&= \Bigg| \left\| V \left(I - |r(j\omega_0)| e_1 e_1' \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \right\| \\
&\quad - \left\| U^* \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \left(I - |r(j\omega_0)| e_1 e_1' \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \right\| \Bigg| \\
&= |(1-\sigma_1)/(1-|r(j\omega_0)|\sigma_1)| \\
&> 1,
\end{aligned}
$$

where it should be noted that $|r(j\omega_0)| \cdot ||T_0(j\omega_0)|| < 1$ is employed which is required for the robust stability in case that each channel gain is $2 (k_i = 2,\ \forall i)$. As shown in the above, $A = (I+L)A_0(I+LA_0)^{-1}$ is not a positive real matrix for L given by (A.1). Therefore, (3d) is necessary. ∎

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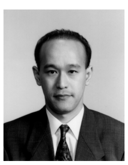

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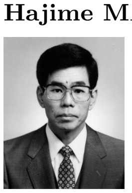

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