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Design of Continuous-Time Deadbeat Tracking Systems

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This paper is concerned with deadbeat control of continuous-time systems. The objective is to construct a continuous-time deadbeat tracking system, in which the tracking error settles down to zero in a finite time or deadbeatly. It is shown that such a tracking system can be obtained by using controllers which include some delay elements.

Key Words: deadbeat control, continuous-time system, delay element

1. Introduction

A dynamical system is called to have the *deadbeat settling property* or *finite settling property* if its transient response vanishes in a finite time. It is desirable for tracking systems to have the deadbeat property, in which the tracking errors vanish completely in a pre-specified time. The objective of this paper is to give a design method for constructing such tracking systems for continuous-time systems.

There are many design methods of deadbeat tracking systems for discrete-time systems, but a few for continuous-time systems. The first one for continuous-time systems is "posicast control" proposed by Smith [3]. The posicast control is open-loop feedforward control for tracking a step function completely in a finite time. The control principle is to superpose some delayed step functions on the step function to be followed so that the tracking error has the deadbeat property. Ando [4] has generalized the posicast control to a general case. Although the idea of the posicast control is interesting the open-loop scheme is lack of generality in the sense that in deadbeat tracking systems it leads to dependence of the controller design on initial states of the plant. Recently, Kurosawa [5] has proposed a different kind of deadbeat tracking scheme for continuous-time systems, in which a delay element is included in a feedback controller. Although the design method doesn't depend on initial states of the plant it is still lack of generality in the sense that the minimum-phase property of the plant is required for internal stability.

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Motivated by the Kurosawa's work, the authors have derived conditions for a continuous-time signal to have the deadbeat property, and proposed a design method for deadbeat regulation systems [1] which can also be applied to non-minimum-phase plants. In the present paper, by extending the idea of the paper [1] to tracking systems it is shown that deadbeat tracking systems can be constructed for general reference signals. To do this, it is first shown that a continuous-time signal which deadbeatly settles to zero is represented as a *finite Laplace transform* in the frequency domain, and sufficient conditions are given for a signal to be represented as a finite Laplace transform in the frequency domain. Controllers used in this paper have the form of the so-called Youla parameterization with the free parameter including delay elements. By using such controllers the problem to construct a deadbeat tracking system is reduced to an interpolation problem for the free parameter, and it is shown that the problem can be solved on the assumption of the so-called *tracking condition*. As a result, it is turned out that a deadbeat tracking system can be constructed in continuous-time systems only on the assumption of the tracking condition.

In this paper, the following notations are used:

$\boldsymbol{R}$: the set of real numbers,

$\boldsymbol{C}$: the set of complex numbers,

X' : the transpose of X,

$\deg_s a(s)$: the degree of $a(s)$ with respect to s,

$\deg_z a(z)$: the degree of $a(z)$ with respect to z.

1.1 Problem Statement

Consider the following scalar system:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + \boldsymbol{b}u(t)\\ y(t) &= \boldsymbol{c}'\boldsymbol{x}(t)\end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \boldsymbol{R}^n$, $u(t) \in \boldsymbol{R}$, $y(t) \in \boldsymbol{R}$ and $A \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{b} \in \boldsymbol{R}^n$, $\boldsymbol{c} \in \boldsymbol{R}^n$. Suppose $(\boldsymbol{c}', A, \boldsymbol{b})$ is minimal and $P(s)$ denotes the transfer function of the plant:

$$P(s) = \boldsymbol{c}'(sI - A)^{-1}\boldsymbol{b}. \tag{2}$$

In the feedback system in **Fig. 1**, K is a controller and r is a reference signal to be followed. A controller K is said to be *admissible* if K internally stabilizes the feedback system. The reference signal $r(t) \in \boldsymbol{R}$ is supposed to be generated by

$$r(t) = \boldsymbol{c}_r'\boldsymbol{x}_r(t),\ \dot{\boldsymbol{x}}_r = A_r\boldsymbol{x}_r(t) \tag{3}$$

where $\boldsymbol{x}_r(t) \in \boldsymbol{R}^{n_r}$ with $\boldsymbol{x}_r(0) \neq 0$ and $A_r \in \boldsymbol{R}^{n_r \times n_r}$, $\boldsymbol{c}_r \in \boldsymbol{R}^{n_r}$.

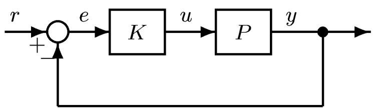

Fig. 1 Feedback system.

Definition. A controller K is called a *deadbeat controller* or K is said to achieve *deadbeat tracking* if K satisfies the following two conditions:

(i) K is admissible.

(ii) The tracking error $e(t) = r(t) - y(t)$ completely settles to zero in a finite time for any initial states $x(0)$ and $x_r(0)$, i.e., for any $\boldsymbol{x}(0)$ and $\boldsymbol{x}_r(0)$, $e(t)$ satisfies

$$e(t) = 0,\ t \geq t_f \tag{4}$$

for some $t_f < \infty$. In this case, t_f is called the *deadbeat settling time.*

Note. In ordinary tracking systems, the tracking error is required to track the reference signal only asymptotically:

$$e(t) \to 0\ (t \to \infty).$$

In this paper, we use the so-called "Youla parameterization" for constructing a deadbeat controller, which is given by

$$\begin{aligned} K(s) &= (M(s)Q(z) - Y(s))(X(s) - N(s)Q(z))^{-1} \\ &= (\tilde{X}(s) - Q(z)\tilde{N}(s))^{-1}(Q(z)\tilde{M}(s) - \tilde{Y}(s)) \end{aligned} \tag{5}$$

where $z := \mathrm{e}^{-sT}$ $(T > 0)$,

$$\begin{bmatrix} \tilde{X}(s) & -\tilde{Y}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} := I + \begin{bmatrix} \boldsymbol{f}' \\ \boldsymbol{c}' \end{bmatrix} (sI - A - \boldsymbol{h}\boldsymbol{c}')^{-1} \begin{bmatrix} -\boldsymbol{b} & \boldsymbol{h} \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} := I + \begin{bmatrix} \boldsymbol{f}' \\ \boldsymbol{c}' \end{bmatrix} (sI - A - \boldsymbol{b}\boldsymbol{f}')^{-1} \begin{bmatrix} \boldsymbol{b} & -\boldsymbol{h} \end{bmatrix} \tag{7}$$

with $\boldsymbol{f} \in \boldsymbol{R}^n$, $\boldsymbol{h} \in \boldsymbol{R}^n$ chosen so that

$$d_f(s) := \det(sI - A - \boldsymbol{b}\boldsymbol{f}'), \tag{8}$$

$$d_h(s) := \det(sI - A - \boldsymbol{h}\boldsymbol{c}') \tag{9}$$

are stable polynomials. In this paper, the free parameter $Q(z)$ is chosen as a polynomial of a delay element $z = \mathrm{e}^{-sT}$ $(T > 0)$:

$$Q(z) = Q_0 + Q_1 z + \cdots + Q_q z^q \tag{10}$$

where $Q_i \in \boldsymbol{R}$ $(i = 0, \cdots, q)$.

Note. If the free parameter Q can be realized causally and stably the controller K of (5) can causally be realized and the closed-loop stability will be guaranteed. In our case, Q of (10) can be realized causally and stably; in fact, the input-output relation

$$w(s) = Q(z)v(s)$$

is realized causally and stably as

$$w(t) = Q_0 v(t) + Q_1 v(t - T) + \cdots + Q_q v(t - qT).$$

A causal realization of K with this $Q(z)$ is given by

$$\begin{aligned} u(t) &= \boldsymbol{f}'\tilde{\boldsymbol{x}}(t) - \sum_{i=0}^{q} Q_i(y(t - iT) - r(t - iT) \\ &\qquad - \boldsymbol{c}'\tilde{\boldsymbol{x}}(t - iT)) \end{aligned} \tag{11}$$

$$\dot{\tilde{\boldsymbol{x}}}(t) = A\tilde{\boldsymbol{x}}(t) + \boldsymbol{b}u(t) - \boldsymbol{h}(y(t) - r(t) - \boldsymbol{c}'\tilde{\boldsymbol{x}}(t)), \tag{12}$$

which is known as an observer-based realization of the Youla parameterization. Hence, the controller K with Q of (10) is admissible, and the problem of constructing a deadbeat controller is reduced to that of finding Q so that the condition (ii) is achieved.

2. Deadbeat Property of Finite Laplace Transform

In this section, it is shown that a continuous-time signal which settles to zero in a finite time is characterized as a finite Laplace transform in the frequency domain.

Let $\phi(s)$ be the Laplace transform of a continuous-time signal $\phi(t)$, i.e.,

$$\phi(s) = \int_0^\infty \phi(t)\mathrm{e}^{-st}dt,$$

and truncate the integral range at $t = T$ to define $\phi_T(s)$ by

$$\phi_T(s) := \int_0^T \phi(t)\mathrm{e}^{-st}dt,$$

which is called the "finite Laplace transform" of $\phi(t)$. The finite Laplace transform $\phi(s)$ can be viewed as the Laplace transform of the truncated signal

$$\phi_T(t) := \begin{cases} \phi(t), & 0 \leq t < T \\ 0, & T \leq t \end{cases},$$

because

$$\phi_T(s) = \int_0^\infty \phi_T(t)\mathrm{e}^{-st}dt.$$

This means that the a continuous-time signal settles to zero in a finite time if it is represented as a finite Laplace transform in the frequency domain.

The next Theorem gives the sufficient conditions for a continuous-time signal to be a finite Laplace transform in the frequency domain.

Theorem 1. Let

$$\psi(s) = \frac{\beta(s,z)}{\alpha(s)},\ z = \mathrm{e}^{-sT}$$

where $\alpha(s)$ is a polynomial in s,

$$\beta(s,z) = \beta_0(s) + \beta_1(s)z + \cdots + \beta_q(s)z^q$$

with $\deg_s \beta_i(s) < \deg_s \alpha(s)$ $(i = 0, \cdots, q)$, and $s_i \in \boldsymbol{C}$ $(i = 1, \cdots, p)$ denote the zeros of $\alpha(s)$ with multiplicities m_i $(i = 1, \cdots, p)$. Then if $\beta(s,z)$ satisfies

$$\frac{d^j}{ds^j}\beta(s_i, \mathrm{e}^{-s_iT}) = 0,\ i = 1, \cdots, p;\ j = 0, \cdots, m_i \quad (13)$$

then $\psi(s)$ is a finite Laplace transform, i.e.,

$$\psi(s) = \int_0^{T_f} \psi_0(t)\mathrm{e}^{-st}dt$$

for some $\psi_0(t)$ with

$$T_f \le qT$$

Note. This Theorem shows that if $\psi(s)$ satisfies (13) it settles to zero with the deadbeat settling time qT, i.e.,

$$\psi(t) = \begin{cases} \psi_0(t), & 0 \le t < qT, \\ 0, & qT \le t. \end{cases}$$

3. Construction of Deadbeat Controller

The objective of this section is to construct a deadbeat controller. To do this, we assume the following condition:

(A1) The poles of $r(s)$ are not zeros of $P(s)$, i.e., there exists no $s \in \boldsymbol{C}$ which satisfies both $d_r(s) = 0$ and $n_0(s) = 0$ where

$$d_r(s) := \det(sI - A_r),\ n_0(s) := \boldsymbol{c}'\mathrm{adj}(sI - A)\boldsymbol{b}.$$

Note. This assumption is known as the "tracking condition," which is inevitable for tracking systems in general.

For initial states $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\boldsymbol{x}_r(0) = \boldsymbol{x}_{r0}$ the feedback system in Fig. 1 is represented in the frequency domain as follows:

$$s\boldsymbol{x}(s) - \boldsymbol{x}_0 = A\boldsymbol{x}(s) + \boldsymbol{b}u(s) \quad (14)$$

$$y(s) = \boldsymbol{c}'\boldsymbol{x}(s) \quad (15)$$

$$u(s) = K(s)e(s) \quad (16)$$

$$s\boldsymbol{x}_r(s) - \boldsymbol{x}_{r0} = A_r\boldsymbol{x}_r(s) \quad (17)$$

$$r(s) = \boldsymbol{c}_r'\boldsymbol{x}_r(s) \quad (18)$$

$$e(s) = r(s) - y(s) \quad (19)$$

From these equations

$$\begin{aligned} y(s) &= \boldsymbol{c}'(sI - A)^{-1}\{\boldsymbol{b}u(s) + \boldsymbol{x}_0\} \\ &= \tilde{M}(s)\tilde{N}(s)K(s)e(s) + \boldsymbol{c}'(sI - A)^{-1}\boldsymbol{x}_0 \end{aligned}$$

and

$$\begin{aligned} e(s) &= \{1 + \tilde{M}^{-1}(s)\tilde{N}(s)K(s)\}^{-1} \\ &\quad \times \{\boldsymbol{c}_r'(sI - A_r)^{-1}\boldsymbol{x}_{r0} - \boldsymbol{c}'(sI - A)^{-1}\boldsymbol{x}_0\} \\ &= \{X(s) - N(s)Q(z)\}\tilde{M}(s) \\ &\quad \times \{\boldsymbol{c}_r'(sI - A_r)^{-1}\boldsymbol{x}_{r0} - \boldsymbol{c}'(sI - A)^{-1}\boldsymbol{x}_0\}. \end{aligned}$$

Hence, $e(s)$ can be written as

$$e(s) = \Phi(s)g(s) \quad (20)$$

where

$$\begin{aligned} \Phi(s) &:= \frac{v_0(s) - n_0(s)Q(z)}{d_f(s)d_h(s)d_r(s)} \\ v_0(s) &:= \det(sI - A - \boldsymbol{b}\boldsymbol{f}') - \boldsymbol{c}'\mathrm{adj}(sI - A - \boldsymbol{b}\boldsymbol{f}')\boldsymbol{h} \\ g(s) &:= d_0(s)\boldsymbol{c}_r'\mathrm{adj}(sI - A_r)\boldsymbol{x}_{r0} \\ &\quad - d_r(s)\boldsymbol{c}'\mathrm{adj}(sI - A)\boldsymbol{x}_0, \\ d_0(s) &:= \det(sI - A). \end{aligned}$$

The objective of deadbeat tracking is to completely settle $e(t)$ down to zero in a finite time for any $\boldsymbol{x}_0$ and $\boldsymbol{x}_{r0}$. For this it suffices that $e(s)$ satisfies the condition of Theorem 1 for any $\boldsymbol{x}_0$ and $\boldsymbol{x}_{r0}$. Since $\boldsymbol{x}_0$ and $\boldsymbol{x}_{r0}$ are included only in $g(s)$ it is obvious that $e(s)$ satisfies it for any $\boldsymbol{x}_0$ and $\boldsymbol{x}_{r0}$ if $\Phi(s)$ satisfies the condition of Theorem 1. Hence, to obtain a deadbeat controller it suffices to find $Q(z)$ such that $\Phi(s)$ satisfies the condition of Theorem 1. In the sequel, it will be shown that such $Q(z)$ can be obtained on the assumption of (A1)

Let

$$\Delta(s) := d_f(s)d_h(s)d_r(s) = \prod_{i=1}^{p}(s - s_i)^{m_i}$$

$(s_i \ne s_j$ if $i \ne j)$ then the condition of Theorem 1 for $\Phi(s)$ becomes

$$\frac{d^j}{ds^j}\{v_0(s_i) - n_0(s_i)Q(\mathrm{e}^{-s_iT})\} = 0, \quad (21)$$
$$i = 1, \cdots, p;\ j = 0, \cdots, m_i - 1.$$

Here, since $(\boldsymbol{c}', A, \boldsymbol{b})$ is minimal $d_f(s)d_h(s)$ can arbitrarily be chosen by appropriate choices of $\boldsymbol{f}$ and $\boldsymbol{h}$, so that it can be assumed that $\Delta(s)$ satisfies the following (B1).

(B1) On the assumption of (A1) $\Delta(s)$ and $n_0(s)$ has no common zeros.

Moreover, the delay duration T can be chosen so that the following (B2) holds:

(B2) For different zeros s_i, s_j $(s_i \ne s_j)$ of $\Delta(s)$

$$\mathrm{e}^{-s_iT} \ne \mathrm{e}^{-s_jT}.$$

It can be shown that on the assumptions (B1) and (B2) there exists $Q(z)$ with $q = 2n + n_r - 1$ which satisfies (21), (The proof will be shown in the appendix.) Hence, using the obtained $Q(z)$ a deadbeat controller can be constructed. In particular, if $\Delta(s)$ has no multiple zeros the condition (21) is represented as the interpolation conditions:

$$Q(\mathrm{e}^{-s_i T}) = \frac{v_0(s_i)}{n_0(s_i)},\ i = 1, \cdots, 2n + n_r. \qquad (22)$$

By the Lagrange interpolation theory $Q(z)$ satisfying (22) is given by

$$Q(z) = \sum_{j=1}^{2n+n_r} \frac{v_0(s_j)}{n_0(s_j)} \frac{L(z)}{(z - z_j)(dL(z_j)/dz)}\bigg|_{z=\mathrm{e}^{-sT}}$$

$$L(z) = (z - z_1)(z - z_2)\cdots(z - z_{2n+n_r}),\ z_j = \mathrm{e}^{-s_j T}.$$

Let's summarize the procedure to obtain a deadbeat controller:

Step 1 Choose $\boldsymbol{f}$, $\boldsymbol{h}$ and T so that (B1) and (B2) are satisfied and $d_f(s)d_h(s)$ is a stable polynomial.

Step 2 Find $Q(z)$ such that the condition (21) is satisfied.

Step 3 Using $\boldsymbol{f}$, $\boldsymbol{h}$ and $Q(z)$ obtained in **Steps 1** and **2** construct a controller $K(s)$ of (5).

Note. 1) Using $K(s)$ obtained above the deadbeat settling time t_f of the tracking error $e(t)$ is

$$t_f \leq (2n + n_r - 1)T.$$

2) The controller $K(s)$ can be represented as

$$K(s) = \frac{w_0(s) + d_0(s)Q(z)}{v_0(s) - n_0(s)Q(z)}$$

where

$$w_0(s) = \boldsymbol{f}'\mathrm{adj}(sI - A - \boldsymbol{b}\boldsymbol{f}')\boldsymbol{h}.$$

The condition (21) implies that zeros of $r(s)$ are also poles of $K(s)$. This means that the so-called "internal model principle" is satisfied in the deadbeat tracking system.

4. Examples

4.1 Example 1

Consider the following first-order system:

$$\dot{x}(t) = -x(t) + u(t)$$
$$y(t) = x(t)$$

with the transfer function $P(s) = 1/(s + 1)$. In this case, $2n + n_r - 1 = 2$. For the step reference $r(s) = 1/s$ a deadbeat controller will be constructed.

Step 1: Let $f = -1$ and $h = -2$ so that $d_f(s)d_h(s) = (s + 2)(s + 3)$, and let $T = 0.5$.

Step 2: $Q(z)$ is obtained as the Lagrange interpolation polynomial:

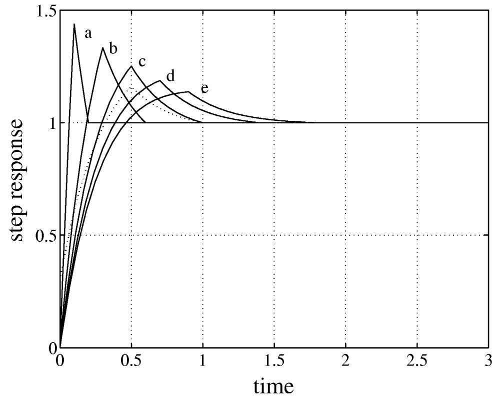

Fig. 2 Simulation results of the example 1.

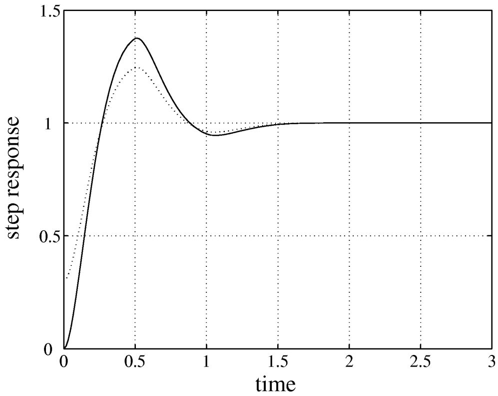

Fig. 3 Simulation results of the example 2.

$$Q(z) = 5.6300 - 1.801z + 0.1714z^2.$$

Step 3: Construct a deadbeat controller as an observer-based realization.

The simulation results are shown in **Fig. 2**. The solid line c is the step response for $x(0) = 0$, and the dotted line is the one for $x(0) = 0.3$. In both cases, the responses settle completely to zero in

$$t = (2n + n_r - 1)T = 1.0.$$

The solid lines a, b, d, e are the step responses for $T = 0.1, 0.3, 0.7, 0.9$, respectively. The corresponding $Q(z)$'s are as follows:

$$\mathrm{a}: Q(z) = 17.3922 - 16.9609z + 3.5687z^2,$$
$$\mathrm{b}: Q(z) = 7.5114 - 4.1034z + 0.5920z^2,$$
$$\mathrm{d}: Q(z) = 4.8874 - 0.9448 + 0.0574z^2,$$
$$\mathrm{e}: Q(z) = 4.5193 - 0.5397z + 0.2040z^2.$$

4.2 Example 2

Consider the following second-order system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = x_1(t)$$

with the transfer function $P(s) = 1/\{(s+1)(s+2)\}$ In this case, $2n + n_r - 1 = 4$. For the step reference $r(s) = 1/s$ a deadbeat controller will be constructed.

Step 1: Let

$$\boldsymbol{f}' = \begin{bmatrix} -10 & -4 \end{bmatrix}, \boldsymbol{h}' = \begin{bmatrix} -8 & -4 \end{bmatrix}$$

so that

$$d_f(s)d_h(s) = (s+3)(s+4)(s+5)(s+6),$$

and let $T = 0.5$.

Step 2: $Q(z)$ can be obtained as the Lagrange interpolation polynomial:

$$\begin{aligned} Q(z) &= 92.2956 - 23.2829z + 3.1771z^2 \\ &\quad - 0.1939z^3 + 0.0042z^4. \end{aligned}$$

Step 3: Construct a deadbeat controller as an observer-based realization.

The simulation results are shown in **Fig. 3**. The solid line is the step response for $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}'$, and the dotted line is the one for $\boldsymbol{x}(0) = \begin{bmatrix} 0.3 & 0 \end{bmatrix}'$. The deadbeat settling time is

$$(2n + n_r - 1)T = 2.0,$$

while the responses settle almost in $t = 1.5$, because the coefficient of z^4 in $Q(z)$ is very small.

5. Concluding Remarks

It has been shown in this paper that a deadbeat tracking system can be constructed even for continuous-time systems by using delay elements in the feedback controller only on the assumption of the tracking condition (A1).

If some model error exists in the plant model the deadbeat tracking would not be obtained. However, as mentioned in the note the deadbeat tracking system satisfies the internal model principle, which implies that at least "asymptotical" tracking will be achieved even if the model error exists.

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Appendix A. Proof of Theorem 1

For $a \in \boldsymbol{C}$ let

$$\theta_a(s, \mathrm{e}^{-sT}) := \int_0^T \mathrm{e}^{at}\mathrm{e}^{-st}dt = \frac{1 - e^{-aT}e^{-st}}{s - a}, \ T > 0$$

then

$$\begin{aligned} \frac{d^k}{ds^k}\theta_a(s, \mathrm{e}^{-st}) &= \int_0^T (-t)^k \mathrm{e}^{at}\mathrm{e}^{-st}dt \\ &= \frac{(-1)^k k!}{(s-a)^{k+1}}\{1 - \mathrm{e}^{aT}\mathrm{e}^{-sT}\sum_{j=0}^{k}\frac{T^j}{j!}(s-a)^j\} \end{aligned}$$

$(k = 0, 1, \cdots)$. Hence, let

$$\theta_a^{(k)}(s, \mathrm{e}^{-st}) := \frac{d^k}{ds^k}\theta_a(s, \mathrm{e}^{-st})$$

then $\theta_a^{(k)}(s, z)$ can be written as

$$\theta_a^{(k)}(s, z) = \frac{g_k(s, z)}{f_k(s)} \qquad \text{(A. 1)}$$

$$g_k(s, z) := (-1)^k k! \left\{1 - \mathrm{e}^{aT} z \sum_{j=0}^{k}\frac{T^j}{j!}(s-a)^j\right\}$$

$$f_k(s) = (s-a)^{k+1}$$

with

$$\frac{d^j}{ds^j}g_k(a, \mathrm{e}^{-sT}) = 0, \ j = 0, \cdots, k,$$

which can be verified directly.

Lemma 1. Suppose $h(s, z)$ is a polynomial in both s and z which satisfies

$$\frac{d^j}{ds^j}h(a, \mathrm{e}^{-aT}) = 0, \ j = 0, \cdots, v-1 \ (a \in \boldsymbol{C}) \qquad \text{(A. 2)}$$

with $\deg_s h(s, z) < v$. Then there exist polynomials (in z) $c_j(z)$ $(j = 0, \cdots, v-1)$ such that

$$\frac{h(s, z)}{(s-a)^v} = \sum_{j=0}^{v-1} c_j(z)\theta_a^{(j)}(s, z) \qquad \text{(A. 3)}$$

$$\deg_z c_j(z) \leq \deg_z h(s, z) - 1. \qquad \text{(A. 4)}$$

Proof. Since $\deg_s h(s, z) < v$ there exists polynomials (in z) $c_i^0(z)$ $(i = 1, \cdots, v)$ such that

$$\frac{h(s, z)}{(s-a)^v} = \frac{c_v^0(z)}{(s-a)^v} + \cdots + \frac{c_1^0(z)}{s-a}.$$

From the condition (A. 2) for $j = 0$

$$c_v^0(\mathrm{e}^{-aT}) = 0,$$

and hence $c_v^0(z)$ can be written as

$$c_v^0(z) = \{c_v^1(z)(-1)^{v-1}(v-1)!\}(1 - \mathrm{e}^{aT}z)$$

where $c_v^1(z)$ is a polynomial in z with

$$\deg_z c_v^1(z) \le \deg_z h(s,z) - 1.$$

Since from (A. 1) $\theta_a^{(v-1)}$ can be written as

$$\theta_a^{(v-1)}(s,z) = \frac{(-1)^{v-1}(v-1)!(1-\mathrm{e}^{aT}z)}{(s-a)^v} + \frac{r_{v-1}(s,z)}{(s-a)^{v-1}}$$

for some $r_{v-1}(s,z)$ which is a polynomial in both s and z we have

$$\frac{h(s,z)}{(s-a)^v} = c_v^1(z)\theta_a^{v-1}(s,z) + \frac{c_{v-1}^1(z)}{(s-a)^{v-1}} + \cdots + \frac{c_1^1(z)}{s-a} \quad \text{(A. 5)}$$

where $c_i^1(z)$ $(i=1,\cdots,v)$ are polynomials in z.
Next, apply the condition (A. 2) for $j=1$ to (A. 5) to have

$$c_{v-1}^1(\mathrm{e}^{-aT}) = 0.$$

By the similar argument above we have

$$\frac{h(s,z)}{(s-a)^v} = c_v^1(z)\theta_a^{(v-1)}(s,z) + c_{v-1}^2(z)\theta_a^{(v-2)}(s,z) + \frac{c_{v-2}^2(z)}{(s-a)^{v-2}} + \cdots + \frac{c_1^2(z)}{s-a}$$

for some $c_i^2(z)$ $(i=1,\cdots,v-1)$ which are polynomials in z.
By repeating the similar argument above until $j=v-1$ we have (A. 3) and (A. 4). ∎

Proof of Theorem 1.

First, decompose $\psi(s,z)$ into the partial fractional decomposition with respect to s as follows:

$$\psi(s,z) = \frac{h_1(s,z)}{(s-s_1)^{m_1}} + \cdots + \frac{h_p(s,z)}{(s-s_p)^{m_p}} \quad \text{(A. 6)}$$

where $h_i(s,z)$ $(i=1,\cdots,p)$ are polynomials in both s and z. Then from (13) $h_i(s,z)$ satisfies

$$\frac{d^j}{ds^j}h_i(s_i,\mathrm{e}^{-s_iT}) = 0,\ j=0,\cdots,m_i-1,$$

and every term on the right-hand side in (A. 6) is strictly proper in s, because $\psi(s,z)$ is so. This implies that $h_i(s,z)/(s-s_i)^{m_i}$ satisfies the condition of Lemma 1. Hence, $\psi(s,z)$ can be represented as

$$\psi(s,z) = \sum_{i=1}^{p}\sum_{j=0}^{m_i-1} c_{ij}(z)\theta_{s_i}^{(j)}(s,z)$$

where $c_{ij}(z)$ are polynomials in z with

$$\deg_z c_{ij}(z) \le \deg_z \beta(s,z) - 1.$$

By letting

$$c_{ij}(z) = \sum_{k=0}^{q-1} c_{ijk}z^k \ (q = \deg_z \beta(s,z))$$

we have the following

$$\psi(s,\mathrm{e}^{-sT}) = \sum_{i=1}^{p}\sum_{j=0}^{m_i-1}\sum_{k=0}^{q-1} c_{ijk}e^{-skT}\theta_{s_i}^{(j)}(s,e^{-sT}) = \int_0^{qT}\psi_0(t)\mathrm{e}^{-st}dt$$

where

$$\psi_0(t)\begin{cases} = \sum_{i=1}^p\sum_{j=0}^{m_i-1} c_{ijk}(kT-t)^j\mathrm{e}^{s_i(t-kT)}, \\ kT \le t < (k+1)T;\ k=0,\cdots,q-1 \\ = 0,\ qT \le t. \end{cases}$$

This shows $\psi(s)$ is a finite Laplace transform of $\psi_0(t)$. ∎

Appendix B. Proof for Existence of $Q(z)$

Let

$$v_0^{(j)} = \frac{d^jv_0(s)}{ds^j},\ n_0^{(j)} = \frac{d^jn_0(s)}{ds^j},$$

$$Q^{(j)}(\mathrm{e}^{-sT}) = \frac{d^jQ(\mathrm{e}^{-sT})}{ds^j},\ \tilde{Q}^{(j)}(z) = \frac{d^jQ(z)}{dz^j}$$

then

$$Q^{(j)}(e^{-sT}) = (-T)^j\tilde{Q}^{(j)}(z)\big|_{z=\mathrm{e}^{-sT}}.$$

Hence, the condition (21) can be represented as

$$V_e - N_e\Psi\tilde{Q}_e = 0 \quad \text{(B. 1)}$$

where

$$V_e := \begin{bmatrix} V_{e1} \\ \vdots \\ V_{ep}\end{bmatrix},\ V_{ei} := \begin{bmatrix} v_0(s_i) \\ \vdots \\ v_0^{m_i-1}(s_i)\end{bmatrix},$$

$$N_e := \begin{bmatrix} N_{e1} & & 0 \\ & \ddots & \\ 0 & & N_{ep}\end{bmatrix},$$

$$N_{ei} := \begin{bmatrix} n_0(s_i) & & 0 \\ \vdots & \ddots & \\ n_0^{(m_i-1)}(s_i) & \cdots & n_0(s_i)\end{bmatrix},$$

$$\tilde{Q}_e := \begin{bmatrix} \tilde{Q}_{e1} \\ \vdots \\ \tilde{Q}_{ep}\end{bmatrix},\ \tilde{Q}_{ei} := \begin{bmatrix} \tilde{Q}_0(\mathrm{e}^{-s_iT}) \\ \vdots \\ \tilde{Q}_0^{m_i-1}(\mathrm{e}^{-s_iT})\end{bmatrix},$$

$$\Psi := \begin{bmatrix} \Psi_1 & & 0 \\ & \ddots & \\ 0 & & \Psi_p\end{bmatrix},$$

$$\Psi_i := \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & (-T)^{m_i-1}\end{bmatrix}.$$

Now, let

$$Q(z) = Q_0 + Q_1z + \cdots + Q_Mz^M \ (M = 2n + n_r - 1)$$

then

$$\tilde{Q}_e = Z \begin{bmatrix} Q_0 \\ \vdots \\ Q_M \end{bmatrix}, Z := \begin{bmatrix} Z_1 \\ \vdots \\ Z_p \end{bmatrix} \qquad \text{(B. 2)}$$

where

$$Z_i := \begin{bmatrix} 1 & z_i & z_i^2 & \cdots & \cdots & z_i^M \\ & \ddots & \ddots & & & \\ 0 & & 1 & z_i & \cdots & z_i^{M-m_i+1} \end{bmatrix}$$

($z_i := \mathrm{e}^{-s_i T}$). It is known that Z is a generalized Vandermonde matrix and nonsingular if $z_i \neq z_j$ for $i \neq j$. Hence, Z is nonsingular from the assumption of (B2). Moreover, N_0 is also nonsingular from the assumption of (B1), so that we have

$$\begin{bmatrix} Q_0 \\ \vdots \\ Q_M \end{bmatrix} = Z^{-1}\Psi^{-1}N_e^{-1}V_e.$$

This gives $Q(z)$ of degree $M(=2n+n_r-1)$ which satisfies (21). □

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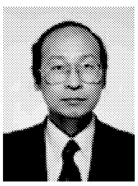

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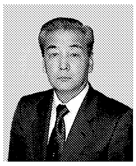

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