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Robust Controller Design Using Parametrization of Stabilizing State Feedback Gains

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This paper proposes a static state feedback controller design method for linear plants that are subject to polytopic uncertainty. Design specifications considered are robust $\mathbf{H}_2$-type disturbance attenuation and/or robust $\mathbf{H}_\infty$-norm constraint with quadratic stabilization.

First, a new parametrization of stabilizing state feedback gains and its geometric structure are investigated. Next, conditions for quadratic stabilization and the robust performances are characterized by certain functions on the parameter space. By optimizing these objective functions, we can obtain state feedback controllers that satisfy prescribed design specifications. Since the parameter space and the functions are convex, the use of suitable convex optimization algorithms assures us they converge to global optimum.

This design method gives a realistic and efficient way for design problems which are difficult to solve analytically. Further, by optimizing the maximum of the objective functions, multiple specifications can be satisfied.

Key Words: parametrization of stabilizing state feedback gains, quadratic stabilization, robust performance, polytopic uncertainty, convex optimization

1. Introduction

Synthesis of the control systems in the real physical world is never able to avoid the influence of uncertainties owing to plant perturbations, modeling and identification errors, noise contaminations and so on. Hence, robust stabilization and robust disturbance attenuation have been recognized as important objectives of control systems synthesis.

The problem we study is synthesis of state feedback gains which simultaneously and robustly achieve stabilization of plants with *structured* parameter uncertainties in the state-space equations, $\mathbf{H}_2$-type disturbance attenuation and $\mathbf{H}_\infty$-norm constraints for closed-loop transfer functions. The structured parameter uncertainties considered here are the polytopic uncertainties[2)3)4)] of the coefficient matrices in the state-space equations. While this uncertainty model can precisely represent real physical parameter uncertainties[1)], it is difficult to derive analytical results[2)3)4)].

This paper proposes the following approach to solve the above problem using convex optimization technique:

[Step1] Parametrize state feedback gains that stabilize a given nominal controlled plant.

[Step2] Represent given control specifications as constraint functions on the parameter space obtained in [Step1].

[Step3] Solve a state feedback gain satisfying all the given specifications by finding a feasible solution via numerical optimization.

The remarkable point is that the parametrization proposed in [Step1] has the following properties:

(i) It parametrizes all the stabilizing state feedback gains.

(ii) The corresponding parameter region is proved to be a convex set.

(iii) The constraint functions in [Step2] are convex on the parameter set.

Hence, the optimization problem in [Step3] turns out convex and the suitable algorithm can find feasible or optimal solutions globally. Further, recent progress of the computational power to perform the optimization is also an important factor to support our numerical approach.

Literature 5)∼8), 16) discuss control systems synthesis on the basis of parametrizations of stabilizing state feedback gains. In 5) and 6) a convex bijective parametrization has been used to derive sufficient conditions for quadratic stabilization and robust $\mathbf{H}_2$- type performance. The other parametrizations for stabilizing state feedback gains are used in 7), 8) and 16). Further, for a restricted class of dynamic output feedback controllers, synthesis methods are proposed using optimization technique[9)10)].

In this paper we first propose a new parametrization by adding redundant parameters to the one given in 5) and 6) and study its geometric structures. Next, utilizing this parametrization we derive a necessary and sufficient condition for quadratic stabilizability by state feedback against the polytopic structured uncertainties. In addition to the quadratic stabilization, we examine a method to synthesize state feedback gains that achieve robust

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performances such as $\mathbf{H}_2$-type disturbance attenuation and $\mathbf{H}_\infty$ norm constraint. For this purpose, we elucidate a necessary and sufficient condition for each $\mathbf{H}_2$-type disturbance attenuation and $\mathbf{H}_\infty$ norm constraint in the nominal case and then develop them to robust case.

The references 7) and 16) address on only the quadratic stabilization problem. While the reference 8) discusses $\mathbf{H}_2$ norm optimization for the plant with structured uncertainties, we consider the wider class of uncertainties. Arguments on another type of structured uncertainties, which is called *norm bounded uncertainties*, with $\mathbf{H}_\infty$ norm constraints can be found in 11).

The advantages of the approach employed in this paper are as follows:

- It gives a realistic and effective way to solve controller synthesis problems that are hard to find solutions in analytic way.
- It enables us to synthesize state feedback gains taking explicitly accounts of multiple design specifications. Further, it is possible to flexibly add new specifications if the new ones are represented by convex constraint functions on the parameter space.

The followings are the organization of this paper. In the section 2, we introduce a new parametrization of stabilizing state feedback gains and investigate geometric structures of the parameter space. In the section 3, we describe design specifications of closed-loop quadratic stability, (robust) $\mathbf{H}_2$ disturbance attenuation and (robust) $\mathbf{H}_\infty$ norm constraints in terms of the parameter space. On the basis of the results, the problem to synthesize state feedback gains satisfying these multiple design specifications is formulated as an optimization problem in the section 4. Finally we demonstrate a numerical design example via the proposed method in the section 5.

2. Parametrization of Stabilizing State Feedback Gains

In this section, we give a parametrization of stabilizing state feedback gains F for the following linear time-invariant plant with n dimensional state vector x and m dimensional input u $(A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m})$:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

(A,B): stabilizable and B: column full rank.

Let $\mathscr{F}_s(A,B)$, $PD(n)$ and $Skew(n)$ denote the set of state feedback gains that stabilize (2.1), the set of n by n symmetric positive definite matrices and the set of n by n skew symmetric matrices, respectively.

2.1 Bijective Parametrization

We introduce the parametrization used in 5) and 6). In this subsection, let Q be an any but fixed positive definite matrix.

Definition.

(i) The set of matrices $P \in PD(n)$ satisfying

$$(I - BB^\dagger)(AP + PA^T + Q)(I - BB^\dagger) = 0. \tag{2.2}$$

is denoted by $PD(n;A,B,Q)$.

(ii) The set of matrices $S \in Skew(n)$ satisfying

$$S = BB^\dagger S BB^\dagger \tag{2.3}$$

is denoted by $Skew(n;B)$.

■

The following theorem gives a parametrization of stabilizing state feedback gains.

Theorem 1. [5,6,12,13] For any $F \in \mathscr{F}_s(A,B)$, there exist $P \in PD(n;A,B,Q)$ and $S \in Skew(n;B)$ that satisfy

$$\begin{aligned} F &= -B^\dagger(AP + PA^T + Q)(I - \frac{1}{2}BB^\dagger)P^{-1} \\ &\quad -B^\dagger S P^{-1}. \end{aligned} \tag{2.4}$$

Conversely, F expressed as (2.4) using any $P \in PD(n;A,B,Q)$ and $S \in Skew(n;B)$ belongs to $\mathscr{F}_s(A,B)$. ■

Note that $F \in \mathscr{F}_s(A,B)$ expressed as (2.4) satisfies Lyapunov equation [5,6,12]

$$(A + BF)P + P(A + BF)^T + Q = 0. \tag{2.5}$$

The parameter space $PD(n;A,B,Q) \times Skew\ (n;B)$ has the following structures:

Theorem 2. [12]

(i) The set $\mathscr{F}_s(A,B)$ is diffeomorphic to the product set $PD(n;A,B,Q) \times Skew(n;B)$.

(ii) The set $PD(n;A,B,Q)$ is an unbounded convex subset in the convex cone $PD(n)$. The dimension of $PD(n;A,B,Q)$ is $N_P \triangleq m(2n - m + 1)/2$.

(iii) The set $Skew(n;B)$ is a linear subspace of $Skew(n)$. The dimension of $Skew(n;B)$ is $N_S \triangleq m(m-1)/2$.

■

2.2 New Parametrization

The parametrization in the previous subsection with the fixed parameter $Q \in PD(n)$ is satisfactory to analyze various properties[12] because it gives diffeomorphic correspondence with stabilizing state feedback gains. However, in the next section, we find regarding Q as one of the parameters convenient in order to represent design specifications on the parameter space. In this subsection, we describe structures of the parameter space and the correspondence of stabilizing state feedback gains for a new parametrization with (Q,P,S) as parameters.

Definition.

(i) The set $\mathscr{P}(A,B)$ denotes triples of matrices (Q,P,S), where Q and $P \in PD(n)$ satisfy (2.2) and $S \in Skew(n)$.

(ii) Let $\mathbf{F}(Q,P,S)$ denote a map from $\mathscr{P}(A,B)$ to $\mathscr{F}_s(A,B)$:

$$\mathbf{F}(Q,P,S) \triangleq -B^\dagger(AP+PA^T+Q)(I-\frac{1}{2}BB^\dagger)P^{-1} -B^\dagger SP^{-1}. \quad (2.6)$$

■

The following theorem shows the structures of the parameter space $\mathscr{P}(A,B)$:

Theorem 3.

(i) The set $\mathscr{P}(A,B)$ is an $N_Q+N_P+N_S$-dimensional convex cone in $PD(n)\times PD(n)\times Skew(n)$, where $N_Q \triangleq n(n+1)/2$.

(ii) For any $F\in\mathscr{F}_s(A,B)$ and any $Q\in PD(n)$, there exist P and S such that $(Q,P,S)\in\mathscr{P}(A,B)$ and $F=\mathbf{F}(Q,P,S)$.

(iii) For $F\in\mathscr{F}_s(A,B)$ let π_F denote the set of $(Q,P,S)\in\mathscr{P}(A,B)$ that satisfies $F=\mathbf{F}(Q,P,S)$. The set π_F is an $n(n+1)/2$-dimensional convex cone in $\mathscr{P}(A,B)$, which is the intersection of $\mathscr{P}(A,B)$ and a certain linear subspace defined by F.

Proof. The statements (i) and (ii) are obvious. The statement (iii) is obtained by the observation that the parameters $(Q,P,S)\in\pi_F$ should satisfy the linear constraint

$$FP=-B^\dagger(AP+PA^T+Q)(I-\frac{1}{2}BB^\dagger)-B^\dagger S.$$

■

3. Representation of Design Specifications on the Parameter Space

3.1 Plant Description

Consider the following linear system with structured uncertainties:

$$\begin{cases} \dot{x}(t)=A(\theta)x(t)+B(\theta)u(t) \\ \qquad +E_1(\theta)w_1(t)+E_2(\theta)w_2(t) \\ z_i(t)=C_i(\theta)x(t)+D_i(\theta)u(t),\ i=1,2. \end{cases} \quad (3.1)$$

Here, $x(t)\in\mathbf{R}^n$, $u(t)\in\mathbf{R}^m$, $w_i(t)\in\mathbf{R}^{q_i}, i=1,2$ and $z_i(t)\in\mathbf{R}^{p_i}, i=1,2$ denote the state variable, control input, disturbance input and controlled output, respectively. Since we consider possibly different input-output relations of the closed-loop systems with respect to two performance criteria ($\mathbf{H}_2$ and $\mathbf{H}_\infty$), we need to distinguish them by the subscripts 1 and 2. Each matrix $A(\theta),B(\theta),C_i(\theta),D_i(\theta)$ and $E_i(\theta), i=1,2$ is represented with $\theta=(\theta_1,\ldots,\theta_r)$ as follows:

$$\begin{cases} A(\theta)=\sum_{k=1}^r\theta_k\bar{A}_k, \\ B(\theta)=\sum_{k=1}^r\theta_k\bar{B}_k, \\ C_i(\theta)=\sum_{k=1}^r\theta_k\bar{C}_{ik},\ i=1,2, \\ D_i(\theta)=\sum_{k=1}^r\theta_k\bar{D}_{ik},\ i=1,2, \\ E_i(\theta)=\sum_{k=1}^r\theta_k\bar{E}_{ik},\ i=1,2, \end{cases} \quad (3.2)$$

$$\sum_{k=1}^r\theta_k=1,\ \theta_k\ge 0, k=1,\ldots,r. \quad (3.3)$$

We call $\bar{A}_k,\bar{B}_k,\bar{C}_{ik},\bar{D}_{ik}$ and $\bar{E}_{ik}, i=1,2,k=1,\ldots,r$ *vertex matrices* and denote the set of parameters θ satisfying (3.3) by Δ. The form of matrices represented by such as (3.2) and (3.3) is called *matrix polytope*[2)3)]. When $\theta=\theta^0\in\Delta$, the system is regarded as a nominal system and written shortly such as $A(\theta^0)=A$ and so on. The assumption we make for the nominal plant are that

(A,B): stabilizable and B: column full rank.

Let us denote by $G_i^\theta(s)$ the closed-loop transfer functions from w_i to z_i with state feedback $u(t)=Fx(t)$:

$$G_i^\theta(s) \triangleq (C_i(\theta)+D_i(\theta)F)(sI-A(\theta)-B(\theta)F)^{-1}E_i(\theta). \quad (3.4)$$

The transfer functions for the nominal plant is particularly denoted by $G_i(s)$.

3.2 Quadratic Stabilization

Quadratic stability is useful to study robust stability condition for time-varying uncertainties[1)]. While the definition adopted here differs from the usual one[1)] for the sake of simplicity, they are easily found equivalent.

Definition. We say F is a state feedback gain that quadratically stabilizes uncertain systems $\{(A(\theta),B(\theta)) \mid \theta\in\Delta\}$ when there exists $P\in PD(n)$ that satisfies

$$(A(\theta)+B(\theta)F)P+P(A(\theta)+B(\theta)F)^T<0 \quad (3.5)$$

for any $\theta\in\Delta$ ■

Note 1. When (3.5) is satisfied, $V(x)=x^TP^{-1}x$ is a Lyapunov function for any function $\theta(t)\in\Delta$ and initial values of the closed-loop system. ■

The following theorem gives a necessary and sufficient condition, in terms of $\mathscr{P}(A,B)$, for F to quadratically stabilize the uncertain closed-loop.

Theorem 4. State feedback F quadratically stabilizes $\{(A(\theta),B(\theta))\mid\theta\in\Delta\}$ if and only if there exists $(Q,P,S)\in\mathscr{P}(A,B)$ that satisfies

$$F=\mathbf{F}(Q,P,S), \quad (3.6)$$

$$\bar{Q}_k(Q,P,S)>0, k=1,\ldots,r, \quad (3.7)$$

where $\bar{Q}_k(\bullet)$ are defined by

$$\begin{cases} \bar{Q}_k(Q,P,S)\triangleq\bar{M}_k(Q,P,S)+\bar{M}_k^T(Q,P,S), \\ \bar{M}_k(Q,P,S)\triangleq-(\bar{A}_k+\bar{B}_k\mathbf{F}(Q,P,S))P \\ \quad =-\bar{A}_kP+\bar{B}_kB^\dagger(AP+PA^T+Q) \\ \qquad (I-\frac{1}{2}BB^\dagger)+\bar{B}_kB^\dagger S. \end{cases} \quad (3.8)$$

Proof. (Necessity) When F quadratically stabilizes $\{(A(\theta),B(\theta))\mid\theta\in\Delta\}$, there exists, from the definition, $P>0$ that meets (3.5) for any $\theta\in\Delta$. Since it also holds for the nominal system ($\theta=\theta^0$), the matrix Q defined by

$$Q\triangleq-(A+BF)P-P(A+BF)^T \quad (3.9)$$

satisfies $Q \in PD(n)$. Further, for the matrix defined by

$$S \stackrel{\triangle}{=} -BFP - BB^{\dagger}(AP + PA^T + Q)(I - \frac{1}{2}BB^{\dagger}) \qquad (3.10)$$

we have $(Q,P,S) \in \mathscr{P}(A,B)$ and $F = \mathbf{F}(Q,P,S)$[12]. On the other hand, the inequalities (3.5) at the vertex matrices $(\bar{A}_k, \bar{B}_k), k = 1,\dots,r$ become

$$-(\bar{A}_k + \bar{B}_k F)P - P(\bar{A}_k + \bar{B}_k F)^T > 0 \qquad (3.11)$$

for $k = 1,\dots,r$. Substituting $F = \mathbf{F}(Q,P,S)$ into these inequalities, we have (3.7).

(Sufficiency) From (3.7) we have for $\theta \in \Delta$

$$\begin{aligned} &-(A(\theta) + B(\theta)F)P - P(A(\theta) + B(\theta)F)^T \\ &\quad = \sum_{k=1}^{r} \theta_i \{-(\bar{A}_k + \bar{B}_k F)P - P(\bar{A}_k + \bar{B}_k F)^T\} > 0. \end{aligned}$$

■

In the proof, the result that we can arbitrarily select the parameter $Q \in PD(n)$ (Theorem 3, (ii)) is utilized.

Since the functions $\bar{Q}_k(\bullet)$ defined by (3.8) are linear with respect to (Q,P,S), the set of (Q,P,S) that satisfy (3.7) is a convex cone. Hence, the following corollary holds:

Corollary 1. Let T_k; $k = 1,\dots,r$ be any positive semidefinite matrices. State feedback F quadratically stabilizes $\{(A(\theta), B(\theta)) \mid \theta \in \Delta\}$ if and only if there exists $(Q,P,S) \in \mathscr{P}(A,B)$ that satisfies

$$\begin{aligned} &F = \mathbf{F}(Q,P,S), \\ &\bar{Q}_k(Q,P,S) - T_k > 0, k = 1,\dots,r. \end{aligned}$$

Proof. Since the sufficiency is obvious, we prove the necessity. If F quadratically stabilizes $\{(A(\theta), B(\theta)) \mid \theta \in \Delta\}$, Theorem 4 ensures the existence of a certain $(\hat{Q}, \hat{P}, \hat{S}) \in \mathscr{P}(A,B)$ that satisfies $F = \mathbf{F}(\hat{Q}, \hat{P}, \hat{S})$ and $\bar{Q}_k(\hat{Q}, \hat{P}, \hat{S}) > 0$, $k = 1,\dots,r$. Hence, there exists a positive number c such that

$$c\bar{Q}_k(\hat{Q}, \hat{P}, \hat{S}) > T_k, \ k = 1,\dots,r.$$

Define as $Q \stackrel{\triangle}{=} c\hat{Q}$, $P \stackrel{\triangle}{=} c\hat{P}$, $S \stackrel{\triangle}{=} c\hat{S}$. Then from Theorem 3, (iii) and (2.6), we have

$$\begin{aligned} &(Q,P,S) \in \mathscr{P}(A,B), \\ &\mathbf{F}(Q,P,S) = \mathbf{F}(\hat{Q}, \hat{P}, \hat{S}) = F \end{aligned}$$

Hence,

$$\bar{Q}_k(Q,P,S) = c\bar{Q}_k(\hat{Q}, \hat{P}, \hat{S}) > T_k$$

hold for $k = 1,\dots,r$. ■

3.3 $\mathbf{H}_2$-type Robust Disturbance Attenuation

The $\mathbf{H}_2$-type performance criterion of input-output relations theoretically have a connection with the output responses for the impulsive or white noise input and reflect transient response characteristics of systems. In this subsection, we consider $\mathbf{H}_2$-type robust disturbance attenuation by state feedback.

First we consider the nominal system. If $F \in \mathscr{F}_s(A,B)$, then the following Lyapunov equation

$$(A+BF)\tilde{X} + \tilde{X}(A+BF)^T + E_1 E_1^T = 0 \qquad (3.12)$$

has the positive semidefinite solution $\tilde{X}$ uniquely. Define $\tilde{Z}$ as

$$\tilde{Z} \stackrel{\triangle}{=} (C_1 + D_1 F)\tilde{X}(C_1 + D_1 F)^T. \qquad (3.13)$$

Then we can give some $\mathbf{H}_2$-type performance criteria of $G_1(s)$ using $\tilde{Z}$ [14]. For example, the trace of $\tilde{Z}$ is equal to the square of $\mathbf{H}_2$ norm of $G_1(s)$. When the positive semidefinite solution $\tilde{X}$ of (3.12) is the covariance matrix of the state variable, the diagonal elements of $\tilde{Z}$ coincide with variances of each element of output z_1 [13].

Here we give a necessary and sufficient condition, in terms of $\mathscr{P}(A,B)$, for state feedback to simultaneously stabilize the nominal closed-loop system and attenuate each diagonal element of $\tilde{Z}$ lower than a prescribed level. We denote the (i,j)-th component of M and the j-th row by $[M]_{ij}$ and $[M]_{j\bullet}$, respectively.

Theorem 5. For given $\gamma_j > 0, j = 1,\dots,p_1$, state feedback F stabilizes $G_1(s)$ and satisfies $[\tilde{Z}]_{jj} < \gamma_j, j = 1,\dots,p_1$ if and only if there exists $(Q,P,S) \in \mathscr{P}(A,B)$ that satisfies

$$F = \mathbf{F}(Q,P,S), \qquad (3.14)$$

$$Q - E_1 E_1^T > 0, \qquad (3.15)$$

$$\begin{bmatrix} \gamma_j & m_j(Q,P,S) \\ m_j^T(Q,P,S) & P \end{bmatrix} > 0, \qquad (3.16)$$

where

$$m_j(Q,P,S) \stackrel{\triangle}{=} [C_1 P + D_1 W]_{j\bullet}, \qquad (3.17)$$

$$\begin{aligned} W = W(Q,P,S) &\stackrel{\triangle}{=} \mathbf{F}(Q,P,S)P \\ &= -\bar{B}_k B^{\dagger}(AP + PA^T + Q)(I - \frac{1}{2}BB^{\dagger}) \\ &\quad + \bar{B}_k B^{\dagger} S. \end{aligned} \qquad (3.18)$$

Proof. (Necessity) Assume that $G_1(s)$ is stable and $[\tilde{Z}]_{jj} < \gamma_j, j = 1,\dots,p_1$ hold for $\tilde{Z}$ defined by (3.13) using the positive semidefinite solution $\tilde{X}$ of the Lyapunov equation (3.12). Let P be the solution of the following Lyapunov equation

$$(A+BF)P + P(A+BF)^T + E_1 E_1^T + \varepsilon I = 0$$

where $\varepsilon > 0$. Then we have $P > \tilde{X}$. In addition if ε is sufficiently small, then

$$[(C_1 + D_1 F)P(C_1 + D_1 F)^T]_{jj} < \gamma_j, \ j = 1,\dots,p_1$$

hold. Substitute (3.17) and (3.18) into this, then we have

$$\gamma_j - m_j(Q,P,S)P^{-1}m_j^T(Q,P,S) > 0, \quad j = 1,\dots,p_1.$$

From the property of positive definiteness of block matrices, (3.16) follows. Further, set

$$Q \stackrel{\triangle}{=} -(A+BF)P - P(A+BF)^T,$$

then $Q - E_1E_1^T = \varepsilon I > 0$ holds. Defining S as (3.10), we have $(Q,P,S) \in \mathscr{P}(A,B)$ and $F = \mathrm{F}(Q,P,S)$.

(Sufficiency) Stability of $G_1(s)$ is due to (3.14). From (2.5) and (3.15),

$$(A+BF)P + P(A+BF)^T + E_1E_1^T < 0$$

holds. This implies, for the solution $\tilde{X}$ of the Lyapunov equation (3.12), $\tilde{X} < P$. Hence, from (3.16) we have $[\tilde{Z}]_{jj} < \gamma_j$, $j = 1,\ldots,p_1$. ■

Next, we consider robust $\mathbf{H}_2$-type disturbance attenuation under the structured uncertainties. For the rest of this subsection, we assume $\theta \in \Delta$ is a constant vector. When $A(\theta)+B(\theta)F$ is stable, we denote by $\tilde{X}(\theta)$ the positive semidefinite solution of the Lyapunov equation

$$\begin{aligned}(A(\theta)+B(\theta)F)\tilde{X}(\theta)+\tilde{X}(\theta)(A(\theta)+B(\theta)F)^T\\ +E_1(\theta)E_1(\theta)^T = 0,\end{aligned} \tag{3.19}$$

and define $\tilde{Z}(\theta)$ as

$$\tilde{Z}(\theta) \stackrel{\triangle}{=} (C_1(\theta)+D_1(\theta)F)\tilde{X}(\theta)(C_1(\theta)+D_1(\theta)F)^T. \tag{3.20}$$

Lemma 1. Let γ_j, $j = 1,\ldots,p_1$ be positive numbers. If the closed-loop system with state feedback F satisfies the following inequalities:

$$\begin{aligned}(A(\theta)+B(\theta)F)P+P(A(\theta)+B(\theta)F))\\ +E_1(\theta)E_1^T(\theta) < 0,\end{aligned} \tag{3.21}$$

$$\begin{aligned}[(C_1(\theta)+D_1(\theta))P(C_1(\theta)+D_1(\theta))]_{jj} < \gamma_j,\\ j = 1,\ldots,p_1\end{aligned} \tag{3.22}$$

for some $P \in PD(n)$ and any $\theta \in \Delta$, then the closed-loop systems is quadratically stable and

$$[\tilde{Z}(\theta)]_{jj} < \gamma_j, j = 1,\ldots,p_1, {}^\forall\theta \in \Delta \tag{3.23}$$

holds for $\tilde{Z}(\theta)$ defined by (3.19) and (3.20).

Proof. Quadratic stability is trivial. Since (3.21) implies $\tilde{X}(\theta) < P$ for $\tilde{X}(\theta)$ defined by (3.19), we can achieve disturbance attenuation (3.23) using (3.22). ■

The conditions in Lemma 1 is sufficient but not necessary for the closed-loop quadratic stability with $\mathbf{H}_2$-type disturbance attenuation. However, it seems convenient and natural in the sense it can be reduced to convex optimization problem (see the following theorem), in order to achieve both specifications simultaneously.

Theorem 6. State feedback F satisfies the inequalities (3.21) and (3.22) in Lemma 1 if and only if there exists $(Q,P,S) \in \mathscr{P}(A,B)$ that satisfies

$$F = \mathbf{F}(Q,P,S), \tag{3.24}$$

$$\bar{Q}_k(Q,P,S) - \bar{E}_{1k}\bar{E}_{1k}^T > 0,\ k = 1,\ldots,r, \tag{3.25}$$

$$\begin{aligned}\begin{bmatrix} \gamma_j & \bar{m}_{jk}(Q,P,S) \\ \bar{m}_{jk}^T(Q,P,S) & P \end{bmatrix} > 0,\\ k = 1,\ldots,r,\ j = 1,\ldots,p_1,\end{aligned} \tag{3.26}$$

where

$$\bar{m}_{jk}(Q,P,S) \stackrel{\triangle}{=} [\bar{C}_{1k}P + \bar{D}_{1k}W(Q,P,S)]_{j\bullet}. \tag{3.27}$$

Proof. (Necessity) Let (3.21) and (3.22) hold and consider (3.21) for the nominal case, i.e., $\theta = \theta^0$. Then defining Q and S by (3.9) and (3.10), respectively, we obtain $(Q,P,S) \in \mathscr{P}(A,B)$ and $F = \mathbf{F}(Q,P,S)$. Substituting these into (3.21) and (3.22) for the case of vertex (i.e., $\theta_k = 1$ for some k), we have (3.25) and (3.26).

(Sufficiency) Substituting (3.24) into (3.25) and (3.26), then we see

$$\begin{bmatrix} -(\bar{A}_k+\bar{B}_kF)P - P(\bar{A}_k+\bar{B}_kF)^T & \bar{E}_k \\ \bar{E}_k^T & I \end{bmatrix} > 0,$$

$$\begin{bmatrix} \gamma_j & [\bar{C}_{1k}+\bar{D}_{1k}F]_jP \\ P[\bar{C}_{1k}+\bar{D}_{1k}F]_j^T & P \end{bmatrix} > 0$$

for $k = 1,\ldots,r$ and $j = 1,\ldots,p_1$. Hence, (3.21) and (3.22) hold for any $\theta \in \Delta$. ■

Note 2. All the inequalities (3.15) and (3.16) in Theorem 5, and (3.25) and (3.26) in Theorem 6 are common in their form, in which affine matrix functions of (Q,P,S) are positive definite. Hence, the sets of parameters satisfying these inequalities are convex.

Further, when $\gamma \to \infty$, the conditions in Theorem 6 become equivalent to

$$\bar{Q}_k(Q,P,S) - \bar{E}_{1k}\bar{E}_{1k}^T > 0,\ k = 1,\ldots,r.$$

From Corollary 1, these are necessary and sufficient for quadratic stability. This means the conditions in Theorem 6 are not conservative with respect to quadratic stability. ■

3.4 Robust $\mathbf{H}_\infty$ norm constraint

Constraints on $\mathbf{H}_\infty$ norm can describe many closed-loop properties such as input-output relations and robust stability conditions for unstructured uncertainties, depending on the closed-loop transfer functions $G_2(s)$ to be considered. We consider here stabilizing state feedback gains F that achieve $\|G_2(s)\|_\infty < \kappa$ for given $\kappa > 0$. The following is the well-known bounded real lemma:

Lemma 2. [15] The closed-loop transfer function $G_2(s)$ is stable and satisfies $\|G_2\|_\infty < \kappa$ if and only if the following Riccati inequality

$$\begin{aligned}(A+BF)P+P(A+BF)^T\\ +\frac{1}{\kappa^2}P(C_2+D_2F)^T(C_2+D_2F)P+E_2E_2^T < 0\end{aligned} \tag{3.28}$$

has a positive definite solution $P > 0$. ■

The next theorem is derived from Lemma 2.

Theorem 7. State feedback F stabilizes $G_2(s)$ and achieves $\|G_2\|_\infty < \kappa$ if and only if there exists $(Q,P,S) \in \mathscr{P}(A,B)$ that satisfies

$$F = \mathbf{F}(Q,P,S), \tag{3.29}$$

$$\begin{bmatrix} Q - E_2E_2^T & (C_2P + D_2W)^T \\ C_2P + D_2W & \kappa^2 I \end{bmatrix} > 0. \qquad (3.30)$$

Proof. (Necessity) Let $G_2(s)$ be stable and satisfy $||G_2||_\infty < \kappa$. Lemma 2 ensures (3.28) has a positive definite solution P. Using this P, define

$$Q \triangleq -(A+BF)P - P(A+BF)^T,$$

then we have $Q > 0$ from (3.28). Further, set S as (3.10), then we can confirm $(Q,P,S) \in \mathscr{P}(A,B)$ and $F = \mathbf{F}(Q,P,S)$. Substitute these into (3.28), then we obtain

$$Q - \frac{1}{\kappa^2}P(C_2 + D_2F)^T(C_2 + D_2F)P - E_1E_1^T > 0.$$

Hence, (3.30) holds.
(Sufficiency) Proceed in the converse way to the proof of the necessity, then we can show (3.28) from (3.29) and (3.30). ■

Next, we consider the case where the structured uncertainties exist.

Lemma 3. [11] If there exists $P \in PD(n)$ that satisfies the following Riccati inequality

$$\begin{aligned} &-(A(\theta) + B(\theta)F)P - P(A(\theta) + B(\theta)F)^T \\ &-\frac{1}{\kappa^2}P(C_2(\theta) + D_2(\theta)F)^T(C_2(\theta) + D_2(\theta)F)P \\ &-E_2(\theta)E_2^T(\theta) > 0 \qquad (3.31) \end{aligned}$$

for $\theta \in \Delta$, the closed-loop system with state feedback F is quadratically stable and satisfies $||G_2^\theta||_\infty < \kappa$ for $\theta \in \Delta$. ■

The conditions in Lemma 3 are sufficient for quadratic stability with $\mathbf{H}_\infty$ norm constraint. The reference 11) uses a similar condition to Lemma 3 to achieve quadratic stability for a type of structured uncertainties different from that in this paper. Since the conditions in Lemma 3 are convenient and natural to utilize convex optimization technique, we employ them in this paper.

Theorem 8. State feedback F satisfies the inequality (3.31) in Lemma 3 if and only if there exists $(Q,P,S) \in \mathscr{P}(A,B)$ that satisfies

$$F = \mathbf{F}(Q,P,S), \qquad (3.32)$$

$$\begin{bmatrix} \bar{Q}_k(Q,P,S) - \bar{E}_{2k}\bar{E}_{2k}^T & (\bar{C}_{2k}P + \bar{D}_{2k}W)^T \\ \bar{C}_{2k}P + \bar{D}_{2k}W & \kappa^2 I \end{bmatrix} > 0,$$
$$k = 1,\ldots,r. \qquad (3.33)$$

Proof. The proof is similar to that of Theorem 6 and omitted. ■

Note 3. The sets of parameters that satisfy Theorem 7 and 8 are convex sets. In a similar way to Note 2, we can verify the inequalities in Theorem 8 reduce to the necessary and sufficient condition

$$\bar{Q}_k(Q,P,S) - \bar{E}_{2k}\bar{E}_{2k}^T > 0$$

by letting $\kappa \to \infty$. ■

4. Convex Optimization Problem

We formulate the results obtained so far on the state feedback design to convex optimization problems.

4.1 Constraint Functions Corresponding to Each Design Specification

The conditions on parameters corresponding to design specifications are all represented as positive definiteness of matrix functions that are affine with respect to parameters. Let $\Phi_i(Q,P,S)$ denote the matrix affine functions. Note that the matrix inequality $\Phi_i(Q,P,S) > 0$ is equivalent to $\phi_i(Q,P,S) = -\lambda_{\min}(\Phi_i(Q,P,S)) < 0$. Further, multiple matrix inequalities $\Phi_i(Q,P,S) > 0, i = 1,2,\ldots$ are equivalent to a single scalar inequality $\phi(Q,P,S) = \max_i\{-\lambda_{\min}(\Phi_i(Q,P,S))\}$. Both ϕ_i and ϕ are known to be convex functions on (Q,P,S) [6]. We call $\phi(Q,P,S)$ *constraint function*.

We next show constraint functions for quadratic stabilization, robust $\mathbf{H}_2$-type disturbance attenuation and robust $\mathbf{H}_\infty$ norm constraints, respectively. For the other specifications, corresponding constraint functions are constructed similarly.

[1] *Quadratic Stabilization:*

From corollary 1, the constraint function is defined by

$$\phi^{\mathrm{QS}}(Q,P,S) \triangleq \max_{1\le k\le r}\{-\lambda_{\min}(\bar{Q}_k(Q,P,S) - T_k)\}$$

using arbitrary positive semidefinite T_k, $k = 1,\ldots,r$.

[2] *Quadratic Stabilization with Robust $\mathbf{H}_2$-type Disturbance Attenuation:*

From Theorem 6, using the functions defined by

$$\phi_1(Q,P,S) \triangleq \max_{1\le k\le r}\{-\lambda_{\min}(\bar{Q}_k(Q,P,S) - \bar{E}_{1k}\bar{E}_{1k}^T)\}$$

and

$$\phi_2(Q,P,S) \triangleq \max_{\substack{1\le j\le p_1 \\ 1\le k\le r}}\left\{-\lambda_{\min}\begin{bmatrix} \gamma_j & \bar{m}_{jk}(Q,P,S) \\ \bar{m}_{jk}^T(Q,P,S) & P \end{bmatrix}\right\},$$

we can construct the corresponding constraint function as

$$\phi^{\mathrm{R-H_2}}(Q,P,S) \triangleq \max\{\phi_1(Q,P,S), \phi_2(Q,P,S)\}.$$

[3] *Quadratic Stabilization with Robust $\mathbf{H}_\infty$ Norm Performance:*

From Theorem 8, using the function defined by

$$\Phi_k^{\mathrm{R-H_\infty}}(Q,P,S) \triangleq \begin{bmatrix} \bar{Q}_k(Q,P,S) - \bar{E}_{2k}\bar{E}_{2k}^T & (\bar{C}_{2k}P + \bar{D}_{2k}W)^T \\ \bar{C}_{2k}P + \bar{D}_{2k}W & \kappa^2 I \end{bmatrix},$$

we define the corresponding constraint function by

$$\phi^{\mathrm{R-H_\infty}}(Q,P,S) \triangleq \max_{1\le k\le r}\{-\lambda_{\min}\Phi_k^{\mathrm{R-H_\infty}}(Q,P,S)\}.$$

4.2 Constraint Function Arising from $\mathscr{P}(A,B)$

First, the parameter space $\mathscr{P}(A,B)$ has linear structures such as symmetry of Q and P, skew symmetry of S, and linear constraints (2.2) and (2.3) between them. Hence, the parameters (Q,P,S) are represented, using mutually independent scalar variables η_i and ζ_i, as

$$\begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix} = \sum_{i=1}^{N_Q+N_P} \eta_i \begin{bmatrix} \tilde{Q}_i & 0 \\ 0 & \tilde{P}_i \end{bmatrix},$$

$$S = \sum_{i=1}^{N_S} \zeta_i \tilde{S}_i,$$

where $\tilde{Q}_i, \tilde{P}_i$ and $\tilde{S}_i$ are linearly independent basis matrices determined by these linear structures.

Additionally, since Q and P should be positive definite, the constraint function

$$\phi^{QP}(Q,P) \triangleq -\lambda_{\min} \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix}$$

is also necessary in optimization procedure.

4.3 Optimization on $\mathscr{P}(A,B)$

Like this way state feedback design problem is reduced to the convex optimization on $\mathscr{P}(A,B)$. In the numerical example in the section 5 we have used the ellipsoid method[10] as an optimization algorithm. This algorithm needs subgradients of the constraint functions, which are given in the Appendix.

Now consider to solve state feedback gain that achieves simultaneously robust $\mathbf{H}_2$-type and $\mathbf{H}_\infty$ norm constraints. Define two sets in $\mathscr{P}(A,B)$ that satisfy the two specifications correspondingly by

$$\Omega^{\mathrm{R-H_2}} \triangleq \{(Q,P,S) \in \mathscr{P}(A,B) \mid \phi^{\mathrm{R-H_2}}(Q,P,S) < 0\}$$

$$\Omega^{\mathrm{R-H_\infty}} \triangleq \{(Q,P,S) \in \mathscr{P}(A,B) \mid \phi^{\mathrm{R-H_\infty}}(Q,P,S) < 0\}.$$

Note that, from Theorem 3, a state feedback gain F satisfying the conditions in Theorem 6 and those in Theorem 8 exists if and only if

$$\Omega^{\mathrm{R-H_2}} \cap \pi_F \neq \emptyset \ \text{ and } \ \Omega^{\mathrm{R-H_\infty}} \cap \pi_F \neq \emptyset. \tag{4.1}$$

It is not clear whether the problem to find such F reduces to a convex optimization problem or not. However, we can find (Q,P,S), by convex optimization, that satisfies

$$F = \mathbf{F}(Q,P,S), \quad (Q,P,S) \in \Omega^{\mathrm{R-H_2}} \cap \Omega^{\mathrm{R-H_\infty}}, \tag{4.2}$$

which is only a sufficient condition for (4.1).

5. Numerical Example

The controlled plant in this numerical example is so-called two-mass-spring system shown in Fig. 1, where two carts with mass $M_1 = 1$ and $M_2 = 1$ are connected by a spring with stiffness $K = 1$.

The control input (force) is added to the left cart. Assume the values of K and M_1 have uncertainties in both $\pm 10\%$ of their nominal values. Let p_1 and p_2 denote, respectively, the positions of the left and right carts and the state variables be $[x_1\ x_2\ x_3\ x_4] = [p_1\ p_2\ \dot{p}_1\ \dot{p}_2]$. The disturbance input forced to the left and right carts are denoted by w_2 and w_1, respectively. The controlled output z_2 is the position of the right cart ($z_2 = p_2$), while the control input u is included in z_1, i.e., $z_1 = [p_2\ u]$.

Then the state space realization is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -h_1 & h_1 & 0 & 0 \\ h_2 & -h_2 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ h_3 \\ 0 \end{bmatrix},$$

$$E_1^T = [0 \quad 0 \quad 0 \quad 1], E_2^T = [0 \quad 0 \quad 1 \quad 0],$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_2 = [0 \quad 1 \quad 0 \quad 0], D_2 = 0,$$

where

$$(h_1, h_2, h_3) = \left(\frac{K}{M_1}, \frac{K}{M_2}, \frac{1}{M_1}\right) = (1,1,1)$$

for the nominal plant. On the other hand, considering the uncertainties of K and M_1, we can specify four vertex matrices by the corresponding values of (h_1, h_2, h_3), i.e.,

$$(0.82, 0.90, 0.91), \quad (1.00, 1.10, 0.91),$$
$$(1.00, 0.90, 1.11), \quad (1.22, 1.10, 1.11).$$

Design specifications are given as follows:

[1] The closed-loop system is quadratically stabilized.

[2] The matrices $\tilde{Z}(\theta)$ given by (3.20) satisfies

$$[\tilde{Z}(\theta)]_{11} < 0.5, [\tilde{Z}(\theta)]_{22} < 80, \ {}^\forall\theta \in \Delta.$$

[3] $\|G_2^\theta\|_\infty < 1.5, \ {}^\forall\theta \in \Delta.$

Hence, the constraint functions employed are $\phi^{\mathrm{R-H_2}}$, $\phi^{\mathrm{R-H_\infty}}$ in the subsection 4.1 and ϕ^{QP} in the subsection 4.2.

It took 30 minutes for the computation with PRO-MATLAB on Sun Sparc Station IPX. The calculated parameters are

$$Q = \begin{bmatrix} 2.037 & -0.140 & -2.194 & -0.853 \\ -0.140 & 0.197 & 0.268 & -0.079 \\ -2.194 & 0.268 & 9.091 & 0.266 \\ -0.853 & -0.079 & 0.266 & 1.581 \end{bmatrix},$$

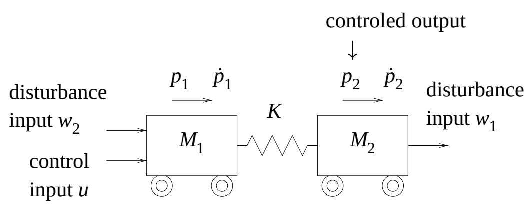

Fig. 1 Two-mass-spring system

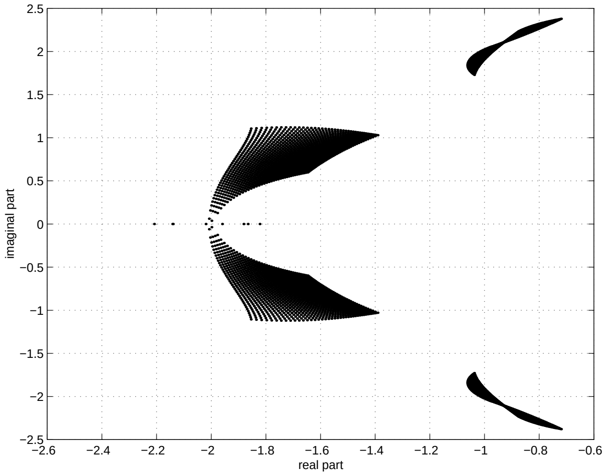

Fig. 2 Closed-loop poles

$$P = \begin{bmatrix} 2.805 & -0.624 & -1.019 & -0.889 \\ -0.624 & 0.498 & 1.029 & -0.099 \\ -1.019 & 1.029 & 12.45 & -2.576 \\ -0.889 & -0.099 & -2.576 & 1.201 \end{bmatrix}.$$

Note that, for scalar input plants like this example, $S = 0$ because $N_S = 0$. Thus, we obtain a state feedback gain

$$F = [-10.68 \quad -4.974 \quad -4.567 \quad -17.28].$$

Fig. 2 shows the plots of the closed-loop poles corresponding to the values of $\theta \in \Delta$ sampled with the equivalent interval. The values of $[\tilde{Z}(\theta)]_{jj}$ for the nominal and four vertex systems are given in Table 1. Fig. 3 and Fig. 4 demonstrate the responses of the controlled output $z_2 = p_2$ and control input u in the case that the unit impulse is input as w_1. Finally, the Bode diagram of $G_2^\theta(s)$ is given in Fig. 5.

Table 1 Values of $\tilde{Z}(\theta)$

	$[\tilde{Z}(\theta)]_{11}$	$[\tilde{Z}(\theta)]_{22}$
nominal	0.352	51.43
vertexes	0.413	59.50
	0.401	41.51
	0.110	40.67
	0.289	40.06

6. Conclusion

In this paper, we propose a new parametrization of stabilizing state feedback gains. Using this, we first elucidate the parameters corresponding to the state feedback gains that quadratically stabilize plants with polytopic structured uncertainties. Next, we also characterize state feedback gains that achieve the performances of $\mathbf{H}_2$-type disturbance attenuation and $\mathbf{H}_\infty$ norm constraints, respectively, in terms of the parameters. Finally, based on these results, we give a method to design state feedback gains that simultaneously and robustly accomplish quadratic stabiliza-

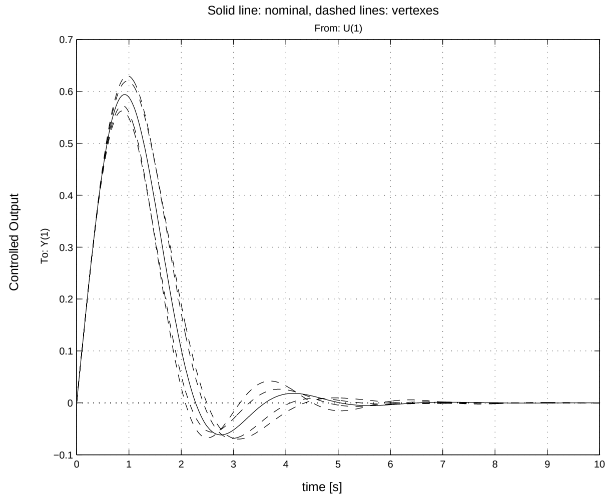

Fig. 3 Impulse responses of control output

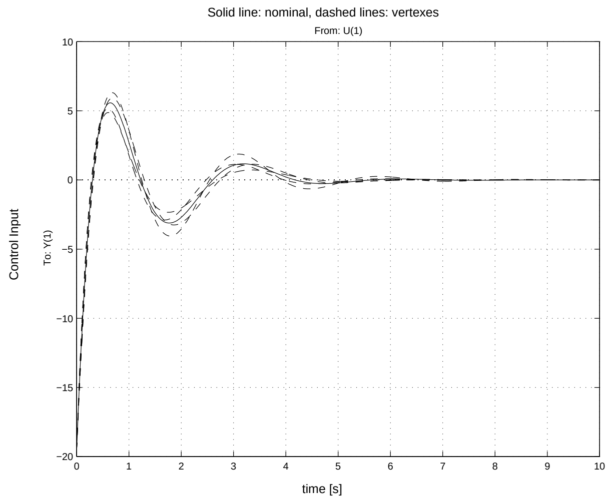

Fig. 4 Impulse responses of control input

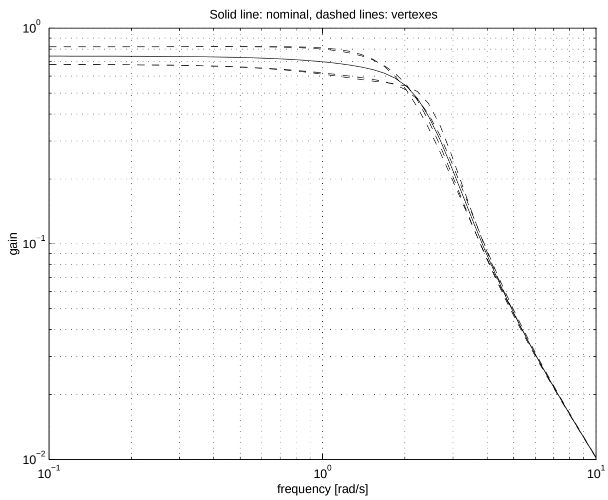

Fig. 5 Bode plots of $G_2^\theta(s)$

tion, $\mathbf{H}_2$-type disturbance attenuation and $\mathbf{H}_\infty$ norm constraints via convex optimization technique. For this purpose, we consider certain robust performance conditions, which seem convenient and natural, and derive their necessary and sufficient conditions characterized by the parameters.

The following items remain for the future work:

- To reduce the conservatism of the proposed method in achievement of multiple specifications, as is discussed in the subsection 4.3, we should develop the design method to find a state feedback gains F that satisfies the condition (4.1).
- We should extend the method to output feedback controller design.

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Appendix

In this appendix, we describe subgradients of the constraint functions in the section 4. Let $\partial\phi$ denote the subgradient[10)] of ϕ. Note that all the constraint functions in the section 4 are represented by the following form

$$\phi = \max_k \{\phi_k(\eta,\zeta)\},\ \phi_k = -\lambda_{\min} H_k(\eta,\zeta),$$

using certain matrices $H_k(\eta,\zeta)$ where $H_k(\eta,\zeta)$ is symmetric matrix-valued affine functions with respect to (η,ζ) and $\eta = (\eta_i) \in \mathbf{R}^{N_Q+N_P}$ and $\zeta = (\zeta_i) \in \mathbf{R}^{N_S}$ are parameters of Q, P and S defined in the subsection 4.2. Let v be the eigenvector corresponding to the minimum eigenvalue of H_k. Then the subgradients of ϕ_k are given by

$$\frac{\partial\phi_k}{\partial\eta_i} = -v^T \frac{\partial H_k}{\partial\eta_i} v$$

and so are those of $\partial\phi_k/\partial\zeta_i$, similarly[6)]. Further, let $\bar{k}$ be the subscript k at (η,ζ) that satisfies $\phi_{\bar{k}}(\eta,\zeta) = \max\{\phi_k(\eta,\zeta)\}$. We have one of the subgradient of ϕ at (η,ζ):

$$\left[\frac{\partial\phi_{\bar{k}}}{\partial\eta}, \frac{\partial\phi_{\bar{k}}}{\partial\zeta}\right] \in \partial\phi.$$

Hence, it is sufficient for us to know $\partial H_k/\partial\eta_i$, which consist of combinations of the partial derivatives of $W(\bullet)$ and $\bar{M}_k(\bullet)$. They are as follows:

$$\begin{aligned}
\frac{\partial W}{\partial\eta_i} &= -B^\dagger(A\tilde{P}_i + \tilde{P}_i A^T + \tilde{Q}_i)(I - \frac{1}{2}BB^\dagger),\\
\frac{\partial W}{\partial\zeta_i} &= -B^\dagger\tilde{S}_i,\\
\frac{\partial\bar{M}_k}{\partial\eta_i} &= -(\bar{A}_k\tilde{P}_i + \bar{B}_k\frac{\partial W}{\partial\eta_i}),\\
\frac{\partial\bar{M}_k}{\partial\zeta_i} &= \bar{B}_k B^\dagger\tilde{S}_i.
\end{aligned}$$

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