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ϵ-Feasibility for $\mathcal{H}_\infty$ Control Problem with Constant Diagonal Scaling†

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This paper considers the $\mathcal{H}_\infty$ control problem with constant diagonal scaling related to the robust control synthesis for systems with structured time-varying uncertainties. It has not been found that the general output feedback problem can be reduced to a convex optimization problem, and hence only a locally optimal solution can be obtained instead of the global one. The purpose of this paper is to provide an algorithm to find a sub-optimal solution with any specified small tolerance or a globally optimal solution for the constantly scaled $\mathcal{H}_\infty$ control problem. We introduce notions of ϵ-feasibility and ϵ-feasibility test algorithm in order to develop desired algorithms. It is shown that we can get an algorithm to find a sub-optimal solution within tolerance $\epsilon > 0$ by combining the bisection method, if we have an ϵ-feasibility test algorithm. The ϵ-feasibility test algorithm named *rectangle covering method* is proposed. We also show that the worst case computational complexity is of polynomial order in the inverse of tolerance and the size of a priori given interval of scaling.

Key Words: robust control synthesis, $\mathcal{H}_\infty$ control, scaling matrix, ϵ-feasibility, global optimization

1. Introduction

Problems of robust stabilization and robust performance synthesis against structured perturbations have been investigated by many researchers, for those problem formulations are more natural than the $\mathcal{H}_\infty$ control setting from the practical application point of view. It is well known that those problems with time-invariant perturbations can be formulated as μ analysis and synthesis problems [2),13)]. However, the μ problems are quite hard to solve, and hence alternative problems, so-called scaled $\mathcal{H}_\infty$ control problems [3),12)], are often used for solving them.

Recently, the relationship between the classes of the perturbations and the corresponding necessary and sufficient conditions for robust stability have been clarified; 1) The constantly scaled $\mathcal{H}_\infty$ norm bound gives the necessary and sufficient condition for arbitrarily fast time-varying perturbations [8),15)]. 2) The dynamically scaled $\mathcal{H}_\infty$ norm bound provides necessary and sufficient condition for time-varying perturbations with restricted rate of variation [16)]. Hence, the scaled $\mathcal{H}_\infty$ control problem can be recognized as one of practically important synthesis problems for robust control design.

In this paper, we consider a constantly scaled $\mathcal{H}_\infty$ control problem, where the class of the scalings is restricted to the class of diagonal matrices. The constantly scaled $\mathcal{H}_\infty$ control problem is often formulated as the following optimization problem: *For a generalized plant, minimize the scaled $\mathcal{H}_\infty$ norm of the closed loop transfer matrix over the stabilizing controllers and the constant diagonal scaling.* For the analysis, i.e., the case where the controller is given, and for the state feedback case, we can see that the problems can be reduced to convex optimization problems involving linear matrix inequality (LMI) conditions [1),4),7)], and hence we can effectively solve them by interior point methods for generalized eigenvalue problem [1)].

However, for the output feedback case, it has not been found that the general constantly scaled $\mathcal{H}_\infty$ control synthesis problem can be reduced to a convex optimization problem. Therefore, only a locally optimal solution can be obtained instead of the global one [3),10),12),17)]. For example, the *D-K* iteration [3),12)], one of the most popular methods to solve the scaled $\mathcal{H}_\infty$ control problem, normally gives a local solution. Also note that a recent approach by bilinear matrix inequalities (BMIs) [6),14)] addresses the problem, but we have not obtained a method for computing the global optimum yet by the BMI approach.

The purpose of this paper is to provide an algorithm to obtain a sub-optimal solution with any specified tolerance. Since the sub-optimal solution tends to the global optimum solution by taking the tolerance smaller, the proposed algorithm gives the global solution. To this end, the notion of ϵ-feasibility is introduced and plays an important role. We will propose an ϵ-feasibility test algorithm,

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which tells us the following: For a given $\epsilon > 0$ and $\gamma\ (> \epsilon)$, γ is either feasible or ϵ-infeasible (i.e., $\gamma - \epsilon$ is infeasible). If we have an ϵ-feasibility test algorithm (see Section 3 for the precise definition), then we can get an algorithm to find a sub-optimal solution with a specified tolerance $\epsilon > 0$ by combining it with a bisection type method. This is the basic idea for developing our algorithm for the global optimization. We shall propose an ϵ-feasibility test algorithms based on point search, namely *rectangle covering method*. We also analyze its computational complexity and show that the worst case computational complexity is of polynomial order in the inverse of tolerance and the size of a priori given interval of scaling.

This paper is organized as follows: We state the problem formulation in Section 2. In Section 3, we introduce the notion of ϵ-feasibility and an ϵ-feasibility test algorithm, and show that the optimization problem is solvable with a specified tolerance based on an ϵ-feasibility test algorithm. Section 5 proposes an ϵ-feasibility test algorithm, and its computational complexity is also analyzed. A concrete algorithm for the optimization problem based on the ϵ-feasibility test algorithm is shown in Section 6. We will give numerical examples to illustrate the proposed algorithm in Section 7. Section 8 offers some concluding remarks.

We will use the following notation: An $n \times m$ matrix with real entries is denoted by $A \in \Re^{n\times m}$, the dimension of a vector a by $\dim(a)$. For a nonnegative definite matrix A, $A^{\frac{1}{2}}$ denotes the unique nonnegative definite square root of A. The linear fractional transformation (LFT) of a 2×2-block matrix G with a matrix K is denoted by $\boldsymbol{F}_l(G,\ K) = G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}$. For a discrete-time or continuous time stable transfer matrix H, the standard $\mathcal{H}_\infty$ norm is denoted by $\|H\|_\infty$.

2. Problem formulation

Consider the robust stabilization problem for the feedback system depicted in **Fig. 1**, where $G(s)$ and $K(s)$ respectively denote the generalized plant and the controller to be designed, and the uncertainty Δ is an element of the norm bounded structured uncertainties:

$$\boldsymbol{\Delta}_s := \{\ \Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_m)\ |\ \|\Delta\| \le 1\ \}$$

We assume without loss of generality that $\dim(e) = \dim(r) = q$. The objective is to find a robustly stabilizing controller $K(s) \in \mathcal{K}_s$ against $\Delta \in \boldsymbol{\Delta}_s$, where $\mathcal{K}_s$ denote the set of proper controllers which internally stabilize the nominal plant $G(s)$.

The robust stabilization problem can be formulated as a so-called scaled $\mathcal{H}_\infty$ control problem [2), 13)]. In this paper, we consider a constantly scaled $\mathcal{H}_\infty$ control problem, where the class of the scaling is restricted to the class of constant diagonal matrices.

The set up is shown in **Fig. 2**, where Σ is a diagonal scaling whose elements are constant. The set of scaling matrices Σ is given by

$$\mathcal{S} := \{\ \mathrm{diag}(\sigma_1 I_{q_1}, \ldots, \sigma_m I_{q_m})\ |\ \sigma_i > 0\ \} \qquad (1)$$

where the dimensions q_i $(i = 1, \ldots, m)$ depend on the uncertainty structure, and they add up to q.

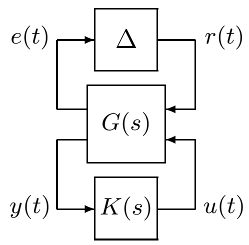

Fig. 1 Uncertain feedback system

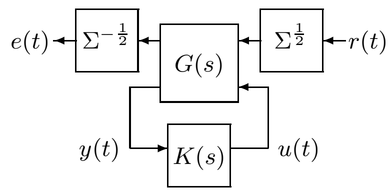

Fig. 2 Constantly scaled $\mathcal{H}_\infty$ control problem

Now, we can define the following feasibility and optimization problems with the scaled $\mathcal{H}_\infty$ norm bound constraint on the closed loop transfer matrix from r to e:

Feasibility Problem (FP):

Given $\gamma > 0$, find $\Sigma \in \mathcal{S}$ such that

$$^{\exists}K(s) \in \mathcal{K}_s;\ \left\|\Sigma^{-\frac{1}{2}}\boldsymbol{F}_l(G,\ K)\Sigma^{\frac{1}{2}}\right\|_\infty < \gamma \qquad (2)$$

Optimization Problem (OP):

Minimize γ subject to the solvability of FP.

Global solutions of the FP and the OP have been obtained for special cases such as state feedback case [9), 11)]. However, in the general output feedback case, the problems have remained open. The difficulty is that we could not re-parameterize the problem so as to make it have desirable properties, such as convexity. The solutions obtained by algorithms in the previous work [3), 10), 12), 17)] only gave the local solutions. The purpose of this paper is to provide an algorithm to obtain a global solution for the

FP and the OP. We will first derive an ϵ-feasibility test algorithm based on point search, then apply the results to the OP.

3. ϵ-feasibility

In this section, we will explain the basic idea of this paper.

To state the idea, we consider the case $m = 2$, i.e., the number of uncertainty blocks is 2, and let $\sigma_2 = 1$ in (1) without loss of generality. In this case, the set $\mathcal{S}$ is given by

$$\mathcal{S} = \{ \ \mathrm{diag}(\sigma I_{q_1}, \ I_{q_2}) \mid \sigma > 0 \ \}$$

Let $\mathcal{F}(\gamma)$ denote the set of solutions of FP defined by

$$\mathcal{F}(\gamma) := \left\{ \ \Sigma \in \mathcal{S} \ \middle| \ {}^{\exists}K \in \mathcal{K}_s; \ \left\| \Sigma^{-\frac{1}{2}} \boldsymbol{F}_l(G, K) \Sigma^{\frac{1}{2}} \right\|_\infty < \gamma \ \right\}$$

Also let γ_{opt} denote the optimal value of the OP given by

$$\gamma_{opt} := \inf_{\mathcal{F}(\gamma) \neq \emptyset} \ \gamma$$

Consider the σ-γ plane shown in **Fig. 3**. We assume that $\mathcal{F}(\gamma) \neq \emptyset$ holds, i.e., the shaded region is nonempty. Since the set $\mathcal{F}(\gamma)$ is not convex in general, there may exist multiple local optimizers as illustrated in Fig. 3. In this case, since the problem has only two parameters (σ and γ), it seems that it is possible to find the optimal value of OP, γ_{opt}, by a mesh type search algorithm. However, we can not obtain the optimal value by the algorithm in general. For example, if the optimal value γ_{opt} is on the sharp point of the boundary shown in Fig. 3, we can not obtain it even if the size of meshes is sufficiently small. In other words, we may need infinitely many search points to find γ_{opt}, and this is not computationally tractable in practice. Thus, it is required to propose another method which enables us to find the global solution.

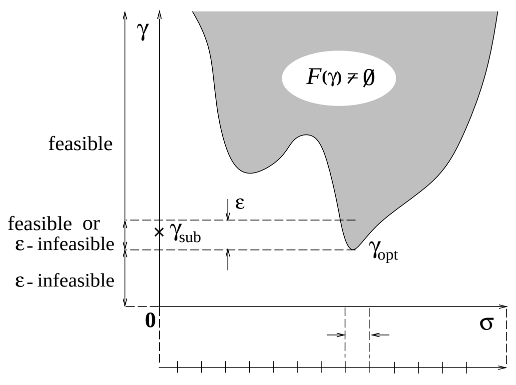

Fig. 3 Feasible region for γ

In this paper, we will not solve the OP directly, but give an algorithm to find a sub-optimal value with given tolerance instead. For this purpose, a notion of the ϵ-feasibility defined below plays an important role.

Definition. For a given $\epsilon > 0$, a given number γ $(> \epsilon)$ is ϵ-infeasible if the FP is not solvable for $\gamma - \epsilon$, i.e., $\gamma - \epsilon$ is infeasible.

Then we can now introduce the ϵ-feasibility test.

Definition (ϵ-feasibility test algorithm). For given $\gamma > 0$ and ϵ s.t. $\gamma > \epsilon > 0$, an algorithm is said to be an ϵ-feasibility test (ϵFT) algorithm if it can conclude that a) γ is feasible or b) γ is ϵ-infeasible, in finite number of steps.

Suppose that we have an ϵFT algorithm. Then, as shown in Fig. 3, the algorithm tells us that,

1) γ is feasible if $\gamma > \gamma_{opt} + \epsilon$.

2) γ is ϵ-infeasible if $\gamma < \gamma_{opt}$.

3) γ is feasible or ϵ-infeasible if $\gamma_{opt} \leq \gamma \leq \gamma_{opt} + \epsilon$.

Hence, by combining the ϵFT algorithm with the bisection method (see Section 6), we can find a sub-optimal value of the OP within tolerance ϵ, e.g., γ_{sub} illustrated in Fig. 3.

In Section 5, we will propose an ϵFT algorithm based on point search. An algorithm for the OP based on ϵFT algorithm will be provided in Section 6.

4. Preliminaries

As we have pointed out in the previous section, the set of solutions of the FP is not convex in general; this makes the problem difficult. One way to isolate the non-convexity of the problem is to use a simple change of variable. In this section, we first derive a necessary and sufficient condition for the solvability of the FP via $\mathcal{H}_\infty$ norm bound condition and several non-convex constraints, and then discuss its properties as a preliminary.

The following is a key lemma in this paper, which gives a necessary and sufficient condition for the solvability of the FP:

Lemma 4.1. Let W_γ and V_γ denote diagonal matrices defined by

$$\begin{aligned} W_\gamma &:= \mathrm{diag}\,(w_1 I, \ldots, w_{m-1} I, \ \gamma I) \in \mathcal{S} \\ V_\gamma &:= \mathrm{diag}\,(v_1 I_{q_1}, \ldots, v_{m-1} I, \ \gamma I) \in \mathcal{S} \end{aligned} \quad (3)$$

For given $\gamma > 0$ and $G(s)$, the FP is solvable if and only if there exist $W_\gamma \in \mathcal{S}$ and $V_\gamma \in \mathcal{S}$ satisfying the following two conditions:

$$^{\exists}K(s) \in \mathcal{K}_s; \ \left\| W_\gamma^{-\frac{1}{2}} \boldsymbol{F}_l(G, \ K) V_\gamma^{-\frac{1}{2}} \right\|_\infty < 1 \quad (4)$$

$$w_i v_i \le \gamma^2, \quad i = 1, \ldots, m-1 \tag{5}$$

Proof: See [10), 19)]. ■

The first condition in Lemma 4.1 is an $\mathcal{H}_\infty$ norm condition. The second condition refers to $m-1$ non-convex constraints, where m is the number of uncertainty blocks. Each of the non-convex constraints, $w_i v_i \le \gamma^2$ $(1 \le i \le m-1)$, defines a non-convex area shown in the w-v plane in **Fig. 4**. If γ is feasible, we see that there exists a solution in the non-convex area. In this paper, this w-v plane plays an important role. In the rest of this paper, we assume that these w_i and v_i are bounded for $^\forall i = 1, \ldots, m-1$, i.e., there exist solutions in the shaded area only, where $\bar{w}_i$ and $\bar{v}_i$ are upper bounds on w_i and v_i such that

$$0 < w_i \le \bar{w}_i < \infty, \quad 0 < v_i \le \bar{v}_i < \infty \qquad i = 1, \ldots, m-1$$

The bounds can be computed by solving the corresponding full information (FI) or full control (FC) problem, since they are convex problems and the feasibility is necessary for that of the original problem.

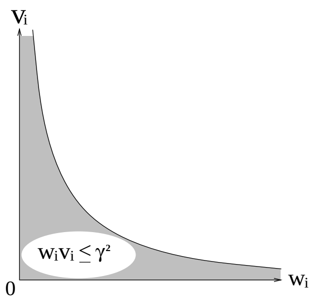

Fig. 4 Non-convex constraint

It is known that the condition (4) in Lemma 4.1 is convex and defined by LMIs [10), 19)], but the condition (5) is not convex. Hence, we can not reduce the FP and OP to successful convex problems in general. However, they have the following desirable properties for our approach:

Lemma 4.2. Let $\mathcal{W}$ denote a set of matrices with $m-1$ repeated scalar block-diagonal elements defined as

$$\mathcal{W} := \{ \operatorname{diag}(w_1 I_{q_1}, \ldots, w_{m-1} I_{q_{m-1}}) \mid w_i > 0, \ i = 1, \ldots, m-1 \} \tag{6}$$

For given $\gamma > 0$, $G(s)$, $W_0 \in \mathcal{W}$ and $V_0 \in \mathcal{W}$ satisfying $W_0 V_0 = \gamma^2 I$, suppose that $W_\gamma = \operatorname{diag}(W_0, \ \gamma I)$ and $V_\gamma = \operatorname{diag}(V_0, \ \gamma I)$ do not satisfy the norm condition (4). Then, there is no pair of $W \in \mathcal{W}$ and $V \in \mathcal{W}$ satisfying (4) with $W_\gamma = \operatorname{diag}(W, \ \gamma I)$ and $V_\gamma = \operatorname{diag}(V, \ \gamma I)$ in the intervals

$$0 < W \le W_0, \quad 0 < V \le V_0$$

Proof: The proof will be done by showing a contradiction.
Suppose that there is no stabilizing $K(s) \in \mathcal{K}_s$ which satisfies (4) for given $\gamma > 0$, W_0 and V_0 satisfying $W_0 V_0 = \gamma^2 I$, and that there exists a set of $K(s) \in \mathcal{K}_s$, $W \in \mathcal{W}$ and $V \in \mathcal{W}$ satisfying the following conditions:

$$\left\| \begin{bmatrix} W & 0 \\ 0 & \gamma I \end{bmatrix}^{-\frac{1}{2}} \boldsymbol{F}_l(G, \ K) \begin{bmatrix} V & 0 \\ 0 & \gamma I \end{bmatrix}^{-\frac{1}{2}} \right\|_\infty < 1 \tag{7}$$

$$0 < W \le W_0, \quad 0 < V \le V_0$$

The norm condition (7) implies that (4) holds for all $\hat{W} \in \mathcal{W}$ and $\hat{V} \in \mathcal{W}$ satisfying $\hat{W} \ge W$ and $\hat{V} \ge V$ with $W_\gamma = \operatorname{diag}(\hat{W}, \ \gamma I)$ and $V_\gamma = \operatorname{diag}(\hat{V}, \ \gamma I)$. Consequently, (4) holds with $W_\gamma = \operatorname{diag}(W_0, \ \gamma I)$ and $V_\gamma = \operatorname{diag}(V_0, \ \gamma I)$, while we have assumed that there is no stabilizing $K(s) \in \mathcal{K}_s$ which satisfies (4). This completes the proof. ■

5. The rectangle covering method

In this section, we will give an ϵFT algorithm. For simplicity, we first show an algorithm for the case $m = 2$, where there is only one non-convex constraint. After that, we will extend the idea to the general case.

5.1 Algorithm

Consider the case $m = 2$. In this case, the matrices W_γ and V_γ in Lemma 4.1 are given by

$$W_\gamma = \operatorname{diag}(wI, \ \gamma I) \in \mathcal{S}, \ V_\gamma = \operatorname{diag}(vI, \ \gamma I) \in \mathcal{S}$$

and the non-convex constraint is given by $wv \le \gamma^2$. The objective of the FP is to find a solution $(w, \ v)$ satisfying the norm condition (4) and the non-convex constraint $wv \le \gamma^2$.

For given $\gamma > 0$ and $G(s)$ with $m = 2$, consider the curve $wv = \gamma^2$ shown in the w-v plane in **Fig. 5**. Let $(w_0, \ v_0)$ denote a point in the w-v plane satisfying $w_0 v_0 = \gamma^2$. If $(w_0, \ v_0)$ is not a solution of the FP, i.e., $(w_0, \ v_0)$ does not satisfy the norm condition (4), then we see from Lemma 4.2 that there is no solution in the shaded rectangular area satisfying

$$0 < w \le w_0, \quad 0 < v \le v_0$$

This implies that we can determine either γ is feasible or there is no solution in the shaded rectangular area. Note that this classification can be done by solving the standard $\mathcal{H}_\infty$ problem [5].

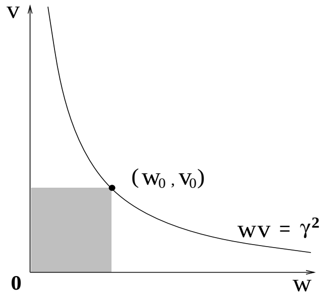

Fig. 5 No solution area

From the above discussions, we have seen that we can determine either γ is feasible or there is no solution in the shaded rectangular area shown in Fig. 5. If we could cover all the non-convex area $wv \le \gamma^2$ by finite number of rectangles, we can exactly determine whether γ is feasible or not by finite steps. However, we need infinitely many points to cover the non-convex area exactly, and this is not computationally tractable in practice. To avoid this, we can do the following.

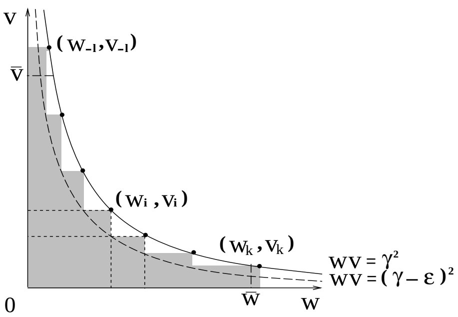

Fig. 6 Rectangle covering method

The idea is to cover a slightly lower curve $wv = (\gamma-\epsilon)^2$, or the area defined by $wv \le (\gamma-\epsilon)^2$, instead of the original non-convex area $wv \le \gamma^2$ by finite number of rectangles as depicted in **Fig. 6**, where ϵ is a given small tolerance satisfying $0 < \epsilon < \gamma$. Let $(w_i,\ v_i),\ i = -k, \ldots, h$ denote finite number of points satisfying $w_i v_i = \gamma^2$. The shaded region is the union of the rectangular areas defined by these points, and it has vertices on both of the upper curve $wv = \gamma^2$ and the lower curve $wv = (\gamma-\epsilon)^2$. The positive integers k and h are determined so that

$$w_{h-1} < \bar{w} < w_h,\ v_{-k+1} < \bar{v} < v_{-k} \tag{8}$$

hold, where $\bar{w}$ and $\bar{v}$ are upper bounds on w and v. If there is no solution in $(w_i,\ v_i),\ i = -k, \ldots, h$, then we see from Lemma 4.2 that there is no solution in the shaded area. Consequently, we can conclude that γ is ϵ-infeasible, since the shaded area covers the lower curve $wv = (\gamma-\epsilon)^2$, or non-convex area $wv \le (\gamma-\epsilon)^2$. Also note that γ is feasible if there exists a solution in $(w_i,\ v_i),\ i = -k, \ldots, h$.

Concretely, these $(w_i,\ v_i)$ are given by the following set of pairs of w_i and v_i:

$$\begin{aligned} \{\ (w_i,\ v_i) \mid w_i &= (1-\eta)^{-2i} w_0 \\ v_i &= (1-\eta)^{2i} v_0,\ i = -k, \ldots, h\ \} \end{aligned} \tag{9}$$

where

$$w_0 = \gamma\sqrt{\bar{w}/\bar{v}},\ v_0 = \gamma\sqrt{\bar{v}/\bar{w}},\ \eta := \epsilon/\gamma \tag{10}$$

We are now in a position to propose an ϵFT algorithm, namely *rectangle covering method*. The following theorem gives the algorithm:

Theorem 5.1. Given $G(s)$ with $m = 2$, $\epsilon > 0$, $\gamma\ (> \epsilon)$, $\bar{w}$, and $\bar{v}$. We can determine γ is either feasible or ϵ-infeasible by the following algorithm:

Rectangle covering method

Initialize: Compute w_0, v_0, and η by (10).

Iteration: Let

$$w_i = (1-\eta)^{-2i} w_0,\quad v_i = (1-\eta)^{2i} v_0$$

$$W_{\gamma,i} = \mathrm{diag}(w_i I,\ \gamma I),\ V_{\gamma,i} = \mathrm{diag}(v_i I,\ \gamma I)$$

and check the following condition for $i = -k, \ldots, h$:

$$\begin{aligned} &{}^{\exists}K(s) \in \mathcal{K}_s \ \ \text{s.t.} \\ &\left\| W_{\gamma,i}^{-\frac{1}{2}} \boldsymbol{F}_l(G,\ K) V_{\gamma,i}^{-\frac{1}{2}} \right\|_\infty < 1 \end{aligned} \tag{11}$$

Stopping criterion: If there exists i satisfying (11), then stop, and we obtain that γ is feasible. Otherwise, γ is ϵ-infeasible.

5.2 Computational complexity analysis

In this subsection, we will discuss about the computational complexity of the algorithm proposed in the previous subsection.

Let N_p denote the number of points required to cover the lower curve $wv = (\gamma-\epsilon)^2$, i.e., $N_p := k + h + 1$, where k and h are defined in (8). In the algorithm, we have to check N_p points to determine that γ is ϵ-infeasible. Hence, the computational complexity can be measured by N_p.

Since the smaller ϵ gives the better precision, N_p grows ϵ gets smaller. N_p also grows as bounds on w and v get larger. To estimate the increase of N_p for these parameters, let us define η and λ by

$$\eta := \epsilon/\gamma,\quad \lambda := \sqrt{\bar{w}\bar{v}}/\gamma \tag{12}$$

The parameter η is a relative tolerance normalized by γ, and λ is a "size" of the scaling parameter space to be sought. By the definition of N_p, we get

$$N_p < -\frac{\ln \lambda}{\ln(1-\eta)} + 1 \tag{13}$$

Notice that the increase in the right hand side of inequality (13) depends on η and λ only. Since

$$-\frac{1}{\ln(1-\eta)} < \frac{1}{\eta} \tag{14}$$

holds for all $0 < \eta < 1$, we can obtain an upper bound on N_p as a polynomial order function for $1/\eta$ and λ:

$$N_p < \ln \lambda \cdot \frac{1}{\eta} + 1 \tag{15}$$

We see that the order of N_p is given by

$$N_p \simeq O\Big(1/\eta \cdot \ln \lambda\Big) \tag{16}$$

This implies that the increase of N_p is proportional to $1/\eta$ and $\ln \lambda$.

5.3 General case

In the previous subsections, we have discussed the case $m = 2$. We will generalize the results for the cases $m \geq 3$, in this subsection.

The algorithm proposed in Theorem 5.1 can be easily extended to the cases $m \geq 3$, where we have two or more non-convex constraints. We only explain an outline of the algorithm for the case $m = 3$. In this case, the matrices $W_\gamma \in \mathcal{S}$ and $V_\gamma \in \mathcal{S}$ in Lemma 4.1 are given by

$$\begin{aligned} W_\gamma &= \mathrm{diag}\,(w_1 I,\ w_2 I,\ \gamma I) \in \mathcal{S} \\ V_\gamma &= \mathrm{diag}\,(v_1 I,\ v_2 I,\ \gamma I) \in \mathcal{S} \end{aligned} \tag{17}$$

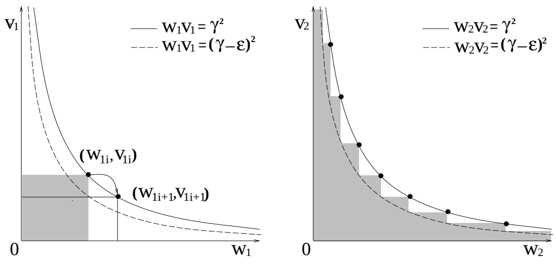

Fig. 7 w_1-v_1 and w_2-v_2 planes

Consider the w_1-v_1 and w_2-v_2 planes shown in **Fig. 7**. What we have to do is to search over the points on the curves $w_1 v_1 = \gamma^2$ and $w_2 v_2 = \gamma^2$. To do that, we fix a point $(w_{1i},\ v_{1i})$ in the w_1-v_1 plane, then apply the rectangle covering method for $m = 2$ to the w_2-v_2 plane. Then, we just repeat this process for different points to cover the lower curves $w_1 v_1 = (\gamma - \epsilon)^2$ and $w_2 v_2 = (\gamma - \epsilon)^2$. Note that these idea can be also generalized for the cases $m \geq 4$ similarly.

For the general case, since the problem has $m - 1$ non-convex constraints, we have to search over the points on $m - 1$ planes. Hence, the order of the number of points N_p for the general case is given as follows:

$$N_p \simeq O\left(1/\eta^{m-1} \cdot \ln \lambda_1 \cdots \ln \lambda_{m-1}\right) \tag{18}$$

where η is defined in (12), and

$$\lambda_i = \frac{\sqrt{\bar{w}_i \bar{v}_i}}{\gamma}, \quad i = 1, \ldots, m-1 \tag{19}$$

We see that the worst case computational complexity of the algorithm is polynomial order in $1/\eta$ and λ_i ($i = 1, \ldots, m-1$).

6. Algorithm for the optimization problem

The purpose of this section is to apply the results described in the previous sections to the OP. As described in Section 3, we can readily obtain an algorithm for the OP by combining the bisection method. Let δ denote a relative tolerance for γ related to a stopping criterion in the bisection method. Then, the algorithm is given as follows:

Optimization algorithm

Given: γ_0 s.t. the FP is solvable, and η satisfying $0 < \eta < 1$.

Initialize: Let $\bar{\gamma} \leftarrow \gamma_0$ and $\underline{\gamma} \leftarrow 0$.

Iteration: Let $\gamma \leftarrow \frac{\bar{\gamma}+\underline{\gamma}}{2}$ and $\epsilon = \eta \cdot \gamma$, and apply the ϵFT algorithm.

case 1. If γ is feasible, $\bar{\gamma} \leftarrow \gamma$.

case 2. If γ is ϵ-infeasible, $\underline{\gamma} \leftarrow \gamma$.

Stopping Criterion: $\bar{\gamma} - \underline{\gamma} < \delta \cdot \bar{\gamma}$ holds for given $0 < \delta < 1$.

In the above algorithm, if $\bar{\gamma} - \underline{\gamma} < \delta \cdot \bar{\gamma}$ holds, then $\gamma_{sub} := \bar{\gamma}$ gives a sub-optimal value with relative tolerance $\hat{\eta} = \eta + \delta - \eta \cdot \delta$. Note that we can apply the algorithm if we have an ϵFT algorithm.

We now analyze the computational complexity of the optimization algorithm combined with the rectangle covering method. Let k denote the number of iterations in the above algorithm. We have

$$\gamma_0 \left(\frac{1}{2}\right)^k < \delta \cdot \gamma_{sub} < \gamma_0 \left(\frac{1}{2}\right)^{k-1}$$

Then, the following inequality holds:

$$k < \frac{\ln(1/\delta) + \ln \rho}{\ln 2} + 1, \quad \rho := \frac{\gamma_0}{\gamma_{sub}}$$

Thus, the order of the worst case computational complexity of the algorithm for the OP is given as follows:

$$O\Big(\ln(1/\delta) \cdot 1/\eta^{m-1} \cdot \ln \lambda_1 \cdots \ln \lambda_{m-1}\Big)$$

where λ_i ($i = 1, \ldots, m-1$) is defined in (19).

7. Numerical examples

In this section, we will provide numerical examples for a system given by $P(s) = 10/s(\tau s + 1)$ to illustrate our proposed algorithm. A state space realization of $P(s)$ is given by the following descriptor form:

$$\begin{bmatrix} 1 & 0 \\ 0 & \tau \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \end{bmatrix} (u + \beta_d d)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + \beta_v v$$

where the process and measurement noises are denoted by d and v with the noise to signal ratios β_d and β_v. The control objective is to regulate the output of the plant, without excessive actuator power, in the presence of process and measurement noises, against uncertainty in the parameter τ. Hence, we set the controlled output as

$$e = \begin{bmatrix} \beta_x \begin{bmatrix} 1 & 0 \end{bmatrix} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \beta_u \end{bmatrix} u$$

with appropriate weights β_x and β_u, and let

$$r = \begin{bmatrix} d & v \end{bmatrix}^T$$

We suppose that there exists a uncertainty in τ, which is given by

$$\tau = \tau_0 + \Delta_\tau, \quad |\Delta_\tau| \le \delta_\tau$$

Hence, the generalized plant $G(s)$ is expressed as

$$G(s) = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1/\tau_0 & -\delta_\tau/\tau_0 & 10 \cdot \beta_d/\tau_0 & 0 & 10/\tau_0 \\ \hline 0 & -1/\tau_0 & -\delta_\tau/\tau_0 & 10 \cdot \beta_d/\tau_0 & 0 & 10/\tau_0 \\ 0 & 0 & 0 & 0 & 0 & \beta_u \\ \beta_x & 0 & 0 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & \beta_v & 0 \end{array}\right]$$

and the matrices W_γ and V_γ in Lemma 4.1 are given by $W_\gamma = \mathrm{diag}(w,\ \gamma I_2) \in \mathcal{S}$ and $V_\gamma = \mathrm{diag}(v,\ \gamma I_2) \in \mathcal{S}$. We have chosen the following nominal values for the plant and disturbance model:

$$\tau_0 = 1, \quad k = 10, \quad \beta_d = 1, \quad \beta_v = 0.1$$
$$\beta_x = 0.189, \quad \beta_u = 0.943$$

We now solve the OP by the optimization algorithm combined with the rectangle covering method. All the computations in this section were carried out by using Xmath. At first, we compute upper bounds on $(w,\ v)$. For a given $\gamma = \gamma_0 > 0$, these bounds can be computed based on a necessary condition for the solvability of the FP. We have chosen the initial value of γ as $\gamma_0 = 4$, and obtained upper bounds $\bar{w}$ and $\bar{v}$ as

$$\bar{w} = 1.53 \times 10^5, \ \bar{v} = 4.01$$

Let N_p denote the number of iterations required to obtain a sub-optimal value, i.e., we have to solve the $\mathcal{H}_\infty$ problem for fixed $(w_i,\ v_i)$ N_p times. We here assume that $\eta = \delta$ in the optimization algorithm proposed in Section 6. Then the relative tolerance for the sub-optimal value of the OP is given by $\hat{\eta} = \eta + \delta - \eta \cdot \delta = 2\eta - \eta^2$. **Fig. 8** plots the number of iterations N_p vs. the relative tolerance $\hat{\eta}$. The solid line refers to the actual case, and the dotted to the worst case. Note that, in Fig. 8, the order of N_p in the worst case is given by $O\left(\ln\left(1/\hat{\eta}\right) \cdot 1/\hat{\eta}\right)$. We see that the increase in N_p is actually less than that for the worst case.

The progress of the sub-optimal value γ_{sub} (upper) vs. the relative tolerance $\hat{\eta}$ (tol) is shown in the solid line in **Fig. 9**, where the dashed line is the lower bound $\underline{\gamma}$. Note that the sub-optimal value γ_{sub} does not always decrease as $\hat{\eta}$ gets smaller, but we can guarantee the global optimality of the solution within relative error $\hat{\eta}$.

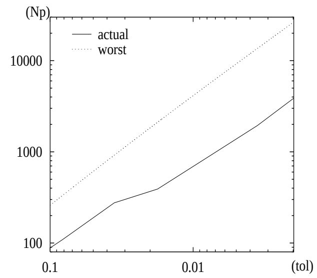

Fig. 8 Number of iterations N_p vs. tolerance $\hat{\eta}$

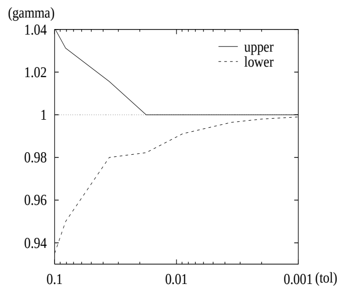

Fig. 9 Upper and lower bounds on γ vs. tolerance $\hat{\eta}$

8. Conclusion

In this paper, we have proposed an algorithm to find a global solution with any specified tolerance for the $\mathcal{H}_\infty$ control problem with constant diagonal scaling. An ϵ-feasibility test algorithm named rectangle covering has been provided, and it has been shown that we can get an optimization algorithm based on the ϵ-feasibility test algorithm. The computational complexity of the algorithm has been also analyzed, and we have shown that its order is polynomial in the inverse of tolerance and the size of a priori given interval of scaling. These properties have been confirmed by numerical examples. We proposed the more efficient algorithm in [18] based on LMI conditions.

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