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Computation Method for Optimal Control Problem with Terminal Constraints using Genetic Algorithm

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In this paper, a new numerical computation method using the genetic algorithm for an optimal control problem with terminal constraints and singular arcs is proposed. The input functions are parameterized using spline interpolation, which has devices that can represent discontinuous input functions. In order to treat the terminal constraints properly, Lagrange multipliers that are contained in genetic information of chromosomes are introduced. On the singular arcs, coefficients of inputs in the Hamiltonian vanish, so the coefficients on the arcs are included in the extended performance index. The weighting coeffecients of the extended performance index are changed adaptively at every generation of the genetic algorithm. A simple example is solved using this method, which verifies the efficiency of the genetic algorithm in the computation of optimal control.

Key Words: genetic algorithm, optimal control, optimal singular control, numerical solution

1. Introduction

In this paper, by using the genetic algorithm, GA, we develop a new numerical method to solve the optimal control problem with terminal constraints, and especially including singular arcs.

When the equations describing the plant and the integrand of the performance index are affine with respect to inputs, the Hamiltonian is also affine with respect to inputs, and thus the optimal inputs during a certain interval of time when the coefficients of inputs in the Hamiltonian are zero cannot be determined using the minimum principle. Such a case is called singular case, and the optimal control problems with the possibility of the appearance of the optimal singular control have not been solved theoretically except in several particular cases. From the point of view of the numerical solution, since the Hessian matrix $H_{uu}(\cdot)$ of the Hamiltonian with respect to inputs becomes singular, all the numerical methods using the regularity of the Hessian matrix cannot apply to the singular optimal control problems. For examples of numerical methods that are applicable to singular cases, we can refer to studies 1)∼3) that require some a-priori information. Moreover, the ϵ-algorithm of Jacobson et al., and the ϵ-$\alpha(\cdot)$-algorithm[4)] that is an improved version of the ϵ-algorithm can be applied to the singular control. In the ϵ-$\alpha(\cdot)$-algorithm, a small term of quadratic form of inputs is added to the Hamiltonian. This algorithm can solve the optimal singular control problem by making the coefficients of the added term tend toward zero. A similar concept is adopted in the method using the convergence control parameter (CCP) of Järmark[5)] and the method of Sakawa and Shindo[6), 7)], which also uses CCP. Moreover, an ϵ-algorithm-like method is used in the method of Chen and Huang[8)]. However, these numerical methods cannot guarantee that global solutions for problems having multi-peaks can be obtained.

With the development of computers, the genetic algorithm is often used for optimization problems in various research areas. The GA is applied to control problems also, for example, studies using the GA to help in the learning of neural networks and studies using the GA for optimization of control design. In this study, the GA is used for obtaining the numerical solutions of optimal control problems with fixed control horizons and terminal state constraints, which include problems having multi-peaks. Seywald et al.[9)] has obtained an optimal control sequence numerically via the GA also, but, in their method, the time is discretized. Therefore, the problem in their method is equivalent to the optimal control problem for discrete-time systems.

In this paper, the data of the input encoded onto chromosomes are not the input sequences themselves. In addition, the data are compressed by using the third spline functions. For the problems with terminal state constraints, Lagrange multipliers with respect to the constraints are introduced, and are encoded onto the chromosomes. To improve the convergence on the singular arc, we add the functions of the coefficients of inputs in

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the Hamiltonian to the performance index. Moreover, the functions of the terminal constraints are used in the extended performance index also, and their weighting coefficients are changed adaptively in every generation of the GA. The effectiveness of this method is confirmed by a simulation result for an example of an optimal control problem for a stirred-tank reactor.

2. Problem statement

We address the optimal control problem of the following system:

$$\dot{x} = f(x, u_2) + \sum_{j=1}^{m_1} g_j(x) u_{1,j} \tag{1}$$

where $x \in \Re^n$ denotes a state vector, and $u \equiv (u_1, u_2) \in \Re^{m_1} \times \Re^{m_2}$ is an input vector. Moreover, it is assumed that $f(\cdot)$ and $g_j(\cdot)$ belong to the $\mathcal{C}^2$-class. We consider the minimizing problem of the performance index

$$J_1 \equiv K(x(t_1)) + \int_{t_0}^{t_1} \left\{ L_0(x(\tau), u_2(\tau)) + \sum_{j=1}^{m_1} L_j(x(\tau)) u_{1,j}(\tau) \right\} d\tau \tag{2}$$

under the constraints

$$x(t_0) = x_0 \tag{3}$$

$$\phi(x(t_1)) \equiv \mathrm{col.}\,(\phi_1(x(t_1)), \ldots, \phi_s(x(t_1))) = 0 \tag{4}$$

$$u_{1,j\,\min}(x(t)) \le u_{1,j}(t) \le u_{1,j\,\max}(x(t)), \quad j = 1, \ldots, m_1;\ t \in [t_0,\, t_1] \tag{5}$$

$$u_{2,j\,\min}(x(t)) \le u_{2,j}(t) \le u_{2,j\,\max}(x(t)), \quad j = 1, \ldots, m_2;\ t \in [t_0,\, t_1] \tag{6}$$

for the system (1), where $u_{i,j\,\min}(x)$, $u_{i,j\,\max}(x)$ $(i = 1, 2, j = 1, \ldots, m_i)$, $K(x)$, and $L_j(x, u_2)$ $(j = 0, 1, \ldots, m_1)$ are $\mathcal{C}^2$ functions. The input vector u_1 appears linearly in the system equation and the performance index, and there is no cross term between u_1 and u_2 in these equations.

We solve this problem by using the GA. An ad hoc way to solve the problem is minimizing

$$J_2 = J_1 + \frac{c}{2} \|\phi(x(t_1))\|^2 \tag{7}$$

instead of J_1, and expecting that the limit solution under $c \to \infty$ satisfies the terminal constraints. However, by this method, it is difficult to keep the error of the terminal constraints within a pre-specified tolerance with a finite constant c, and the objective to minimize the original performance index J_1 may be unachievable. We can employ another performance index[9]

$$J_2' = J_1 + c\|\phi(x(t_1))\|, \tag{8}$$

but the result of this method is very sensitive with respect to the value of parameter c, so that many trials are necessary. In this paper, the performance index is improved by using the co-state and Lagrange multipliers, and a method of changing the weighting coefficients adaptively in every generation of the GA is proposed. In this section, well-known necessary conditions of optimal control will be shown, and, in the next section, we propose an extended performance index including the necessary conditions.

By adding the conditions of constraints, the performance index can be extended to

$$\begin{aligned} J_3 \equiv\ & J_2 + \langle \lambda, \phi(x(t_1)) \rangle \\ & + \int_{t_0}^{t_1} \left\langle p(\tau), f(x(\tau), u_2(\tau)) + \sum_{j=1}^{m_1} g_j(x(\tau)) u_{1,j} - \dot{x} \right\rangle d\tau \\ & + \int_{t_0}^{t_1} \{ \langle \mu_{L1}(\tau), u_{1\min}(x(\tau)) - u_1(\tau) \rangle - \langle \mu_{H1}(\tau), u_{1\max}(x(\tau)) - u_1(\tau) \rangle \} \, d\tau \\ & + \int_{t_0}^{t_1} \{ \langle \mu_{L2}(\tau), u_{2\min}(x(\tau)) - u_2(\tau) \rangle - \langle \mu_{H2}(\tau), u_{2\max}(x(\tau)) - u_2(\tau) \rangle \} \, d\tau, \end{aligned} \tag{9}$$

where $p \equiv \mathrm{row}\,(p_1, \ldots, p_n)$ is the co-state vector, $\lambda \equiv \mathrm{row}(\lambda_1, \ldots, \lambda_s)$ is a Lagrange multiplier vector corresponding to the terminal state constraints, and $\mu_{L1}(t) \equiv \mathrm{row}\,(\mu_{L1,1}(t), \ldots, \mu_{L1,m_1}(t))$, $\mu_{H1}(t) \equiv \mathrm{row}\,(\mu_{H1,1}(t), \ldots, \mu_{H1,m_1}(t))$, $\mu_{L2}(t) \equiv \mathrm{row}\,(\mu_{L2,1}(t), \ldots, \mu_{L2,m_2}(t))$, and $\mu_{H2}(t) \equiv \mathrm{row}\,(\mu_{H2,1}(t), \ldots, \mu_{H2,m_2}(t))$ are Lagrange multipliers corresponding to the lower bound of u_1, the upper bound of u_1, the lower bound of u_2, and the upper bound of u_2 respectively. To simplify the expression, we define

$$\mu \equiv (\mu_{L1}, \mu_{H1}, \mu_{L2}, \mu_{H2}) \tag{10}$$

$$h(x, u) \equiv \begin{pmatrix} u_{1\min}(x) - u_1 \\ u_1 - u_{1\max}(x) \\ u_{2\min}(x) - u_2 \\ u_2 - u_{2\max}(x) \end{pmatrix}. \tag{11}$$

The constraints for the inputs can be expressed as each component of $h(x, u)$ must be less than or equal to zero. The input vector satisfying the condition is called an admissible input. The Hamiltonian is defined as follows:

$$H(x, p, u) \equiv L_0(x, u_2) + \sum_{j=1}^{m_1} L_j(x) u_{1,j} + \left\langle p, f(x, u_2) + \sum_{j=1}^{m_1} g_j(x) u_{1,j} \right\rangle. \tag{12}$$

Then the conditions

$$\dot{p} = -\frac{\partial H}{\partial x} - \mu\frac{\partial h}{\partial x} \tag{13}$$

$$p(t_1) = \left(\frac{\partial K}{\partial x} + (c\phi(x)^T + \lambda)\frac{\partial \phi}{\partial x}\right)\Big|_{x=x(t_1)} \tag{14}$$

$$\frac{\partial H}{\partial u} + \mu\frac{\partial h}{\partial u} = 0 \tag{15}$$

$$\langle \mu, h(x,u)\rangle = 0 \tag{16}$$

$$\mu_j \geq 0, \quad j = 1, \ldots, 2(m_1 + m_2) \tag{17}$$

must be fulfilled along the optimal solution. Moreover, the minimum principle shows that

$$H(x^*, p^*, u^*) \leq H(x^*, p^*, u) \tag{18}$$

for all admissible input u, where u^* denotes the optimal input vector, x^* the optimal state vector, and p^* the corresponding optimal co-state vector. By picking up and gathering the components of $h(x,u)$ corresponding to the active constraints at t, we can define a column vector $\tilde{h}_t(x,u)$, such that

$$\tilde{h}_t(x(t), u(t)) = 0. \tag{19}$$

Let $\tilde{\mu}$ denote a Lagrange multiplier vector of which components correspond to $\tilde{h}_t(x,u)$, then the components of μ excluded by $\tilde{\mu}$ are zero at t. Note that the dimension of $\tilde{h}_t(x,u)$ is not greater than m_1+m_2. From equation (15),

$$\tilde{\mu} = -\left[\frac{\partial \hat{h}_t}{\partial u}\right]^+ \frac{\partial H}{\partial u} \tag{20}$$

is derived. The superscript + means that the matrix is a generalized inverse matrix, but $\tilde{\mu}$ is unique and independent of the choice of the generalized inverse matrix.

Assume that the Hessian matrix

$$\frac{\partial}{\partial u_2}\left[\frac{\partial H}{\partial u_2}\right]^T \tag{21}$$

has full rank, then u_2 minimizing the Hamiltonian is determined uniquely. Conversely, the Hamiltonian is linear with respect to u_1 as follows:

$$H(x,p,u_1,u_2) = H_0(x,p,u_2) + \sum_{j=1}^{m_1} H_j(x,p)u_{1,j}. \tag{22}$$

Therefore, if $H_j(x,p)$ $(j \neq 0)$ is not zero, $u_{1,j}$ has a value on the boundary of the input constraint. However, if $H_j(x,p)$ vanishes on an interval of time, the optimal input cannot be determined by the minimum principle. The optimal trajectory in such an interval is called a singular arc. The optimal control problem with a singular arc has not been solved analytically yet, except in several simple cases. For both inputs, if the value of $u_{i,j}$ is not on the boundary of the constraint in a period,

$$\left.\frac{\partial H}{\partial u_{i,j}}\right|_{x=x^*, p=p^*, u=u^*} = 0 \tag{23}$$

is satisfied for the period.

While the Lagrange multipliers μ's for the input constraints can be obtained with equation (20), it is difficult to solve the Lagrange multipliers λ's for the terminal state constraints explicitly. Hence, in this paper, the information of λ is included in the genes in the GA, and the λ satisfying the above condition and the trajectory fulfilling the terminal constraints are searched through the GA.

3. Fitness function of the GA

Using the necessary conditions on the optimality, we can extend the performance index as

$$\begin{aligned} J_{GA} \equiv J_2 &+ \sum_{i=1}^{s} k_i|\phi_i(x(t_1))| \\ &+ \int_{t_0}^{t_1}\sum_{j=1}^{m_1} a_j\gamma_j(x,p,u)\sqrt{|H_j(x,p)|}\,dt \\ &+ \int_{t_0}^{t_1}\sum_{j=1}^{m_2} b_j|u_{2,j} - \tilde{u}_{2,j}(x,p,u_1)|\,dt \\ &+ q\langle\lambda,\ \phi(x(t_1))\rangle \end{aligned} \tag{24}$$

where $\tilde{u}_{2,j}(x,p,u_1)$ denotes u_2 minimizing the Hamiltonian, and

$$\gamma_j = \begin{cases} 0 & \begin{matrix}(u_{1,j} = u_{1,j\,\min} \textbf{ and } H_j(x,p) > 0) \\ \textbf{or} \\ (u_{1,j} = u_{1,j\,\max} \textbf{ and } H_j(x,p) < 0)\end{matrix} \\ 1 & \textbf{other}. \end{cases} \tag{25}$$

This extended performance index drives the GA procedure. The positive coefficients k_i, a_j and b_j vary adaptively in each generation of the GA. The coefficient q starts from zero, and becomes $q = 1$ after a pre-specified generation. Since the absolute values of the terminal constraints are adopted in the equation of the performance index, the trajectory corresponding to the minimum value of J_{GA} satisfies the terminal constraints for sufficiently large k_i, a_j and b_j. In this study, if the error of the i-th terminal constraint of the best individual in a generation is contained within a tolerance of

$$|\phi_i(x(t_1))| \leq \epsilon_i, \tag{26}$$

the coefficient k_i becomes smaller in the next generation by being multiplied by a constant smaller than 1, and if the error is out of the tolerance, k_i of the next generation becomes large by being multiplied by a constant greater than 1. Moreover, if all the terminal constraints are within the tolerances, a_i $(i = 1, \ldots, m_1)$ and b_i $(i = 1, \ldots, m_2)$ decrease in the next generation, and, if not, these coefficients increase in the next generation. Under the constraints, the minimum values of the four performance indices J_1, J_2, J_3 and J_{GA} coincide with each other as fol-

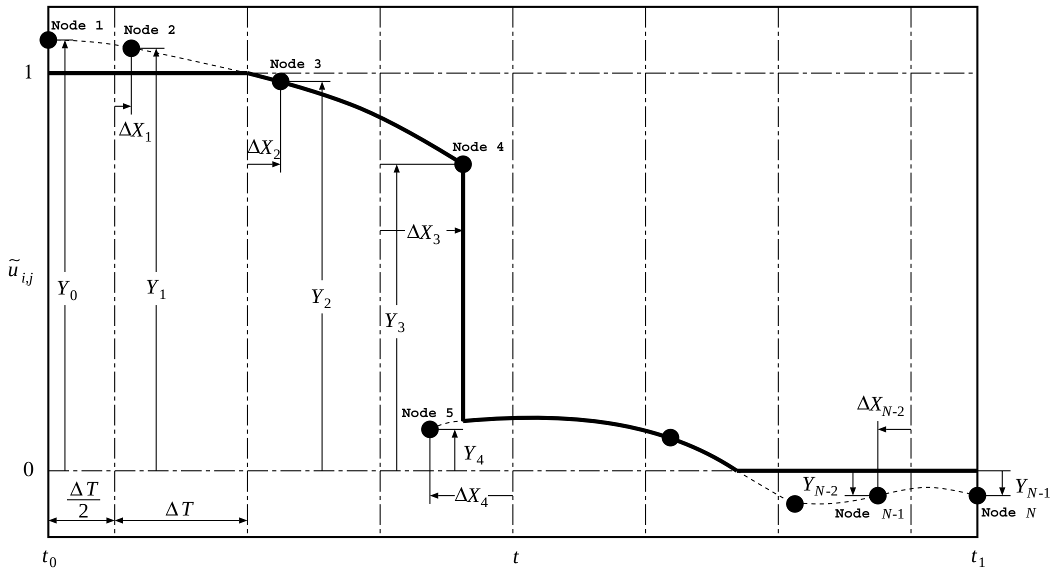

Fig. 1 Parameterization of input.

lows:

$$J_1(x^*, p^*, u^*) = J_2(x^*, p^*, u^*)$$
$$= J_3(x^*, p^*, u^*) = J_{GA}(x^*, p^*, u^*). \quad (27)$$

As the value of J_{GA} is large, the fitness function of the GA should be small. In this paper, we adopt

$$\frac{J_{GA\max} - J_{GA}}{J_{GA\max} - J_{GA\min}} + d \quad (28)$$

as the fitness function, where $J_{GA\min}$ and $J_{GA\max}$ are the minimum value and the maximum value, respectively, of J_{GA} in the generation of the GA, and d is a positive coefficient varying in every generation. For a large d, the values of the fitness function in the generation are leveled evenly, and for a small d, the ratio between the values of the fitness function for two individuals becomes large. If, in the selection stage, the number of the culled individuals is greater than a pre-specified value, d increases in the next generation. Conversely, if the number is smaller than the pre-specified value, d decreases.

4. Coding method of chromosome

The gene in each individual of the GA should contain the information for the sequence of input u and the Lagrange multipliers λ corresponding to the terminal constraints. First, we address the coding method of the input sequence into the chromosome in each individual.

Figure 1 illustrates the coding method for an input sequence. The horizontal-axis shows time, and the vertical-axis shows the input $u_{i,j}$ normalized from $[u_{i,j\min}, u_{i,j\max}]$ into [0, 1]. We place N nodes (node 1, node 2,. . . , node N) at the following coordinates:

$$(t_0, Y_0), \left(t_0 + \frac{\Delta T}{2} + \Delta X_1, Y_1\right),$$
$$\left(t_0 + \frac{3\Delta T}{2} + \Delta X_2, Y_2\right), \ldots \quad (29)$$
$$\ldots, \left(t_1 - \frac{\Delta T}{2} + \Delta X_{N-2}, Y_{N-2}\right), (t_1, Y_{N-1}),$$

where $\Delta T = (t_1 - t_0)/(N - 2)$. The black circles in Fig. 1 indicate the nodes. All nodes are interpolated by a third spline-curve of which the parameter is the time axis. If the value of the normalized input of the interpolated curve overruns the period [0, 1], the curve is clipped within [0, 1]. Moreover, at the point where the values of time of two adjoining nodes are backward, the input sequence becomes discontinuous, and the parts before and after the discontinuous point are interpolated separately. In the example of Fig. 1, there exists a discontinuous point between the 4th node and the 5th node. The chromosome of each individual has the information for the data of the nodes

$$Y_0, \Delta X_1, Y_1, \ldots, \Delta X_{N-2}, Y_{N-2}, Y_{N-1} \quad (30)$$

for all inputs and the data of the Lagrange multipliers $\lambda_1, \ldots, \lambda_s$ corresponding to the terminal constraints. Therefore, the chromosome length is $\ell\{2(N-1)(m_1 + m_2) + s\}$ bits, where every real value is quantized into ℓ bits.

The interpolated inputs, which is denoted by $\tilde{u}_{i,j}(t)$, are normalized into [0, 1]. The real input sequences can be calculated as

$$u_{i,j} = u_{i,j}(x, t) \quad (31)$$
$$= (u_{i,j\max}(x) - u_{i,j\min}(x))\tilde{u}_{i,j}(t) \quad (32)$$

$$+ u_{i,j\,\min}(x), \quad i = 1, 2;\ j = 1, \ldots, m_i,$$

which lead the state vector x and co-state vector p by solving

$$\dot{x} = f(x, u_2(x,t)) + \sum_{j=1}^{m_1} g_j(x, u_2(x,t)) u_{1,j}(x,t) \quad (33)$$

$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, u_1(x,t), u_2(x,t)) + \left[\frac{\partial \hat{h}_t}{\partial u}\right]^+ \frac{\partial H}{\partial u}(x, p, u_1(x,t), u_2(x,t)) \frac{\partial \tilde{h}_t}{\partial x}(x) \quad (34)$$

with a numerical method, e.g., the Runge-Kutta method.

The combination of spline interpolation and the GA is comparatively popular for the optimization problem of a function, but one significant aim of this study is to extend the class of the function to the class of piecewise continuous functions.

5. Example and simulation result

We address the following example:

$$\dot{x} = -(u + 5)x + u \quad (35)$$

$$J_1 = -\int_{t_0}^{t_1} (-4x(\tau) + 3)u(\tau)\, d\tau, \quad (36)$$

where $n = 1$, $m_1 = 1$, $m_2 = 0$, and the inverval of time is $[t_0, t_1]$ ($t_0 = 0$, $t_1 = 1$). The constraint conditions are

$$x(t_0) = 0 \quad (37)$$

$$\phi_1(x(t_1)) = x(t_1) - 0.2 = 0 \quad (38)$$

$$0 \leq u(t) \leq 10. \quad (39)$$

This example is a problem for obtaining the best earnings for the dimension-less model of a stirred-tank reactor, which was studied by Sawaragi et al.[10] The input constraints are independent of the state, therefore, p is independent of μ also.

The constants in the extended performance index J_{GA} are set as follows:

$$c = 4 \quad (40)$$

$$\text{Initial value of } k_1 = 9 \quad (41)$$

$$\text{Initial value of } a_1 = 15. \quad (42)$$

If $|\phi_1(x(t_1))| \leq \epsilon_1 = 0.0025$ for an individual having the largest J_{GA}, which is called the best individual, then a_1 and k_1 are multiplied by 0.997. If $|\phi_1(x(t_1))| > \epsilon_1$ in the best individual, a_1 are k_1 are multiplied by 1.007. The value of q is zero until the 130th generation, and becomes 1 after that generation. In coding chromosome, we set $N = 12$, $\Delta T = (t_1 - t_0)/(N - 2) = 0.1$, $-1.8\Delta T \leq \Delta X_j \leq 1.8\Delta T$, $-0.125 \leq Y_j \leq 1.125$ (20% of this interval is outside of [0, 1]), and $-10 \leq \lambda \leq 10$.

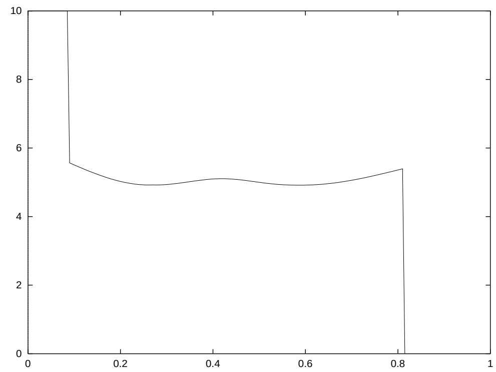

Fig. 2 Optimal input $u_1(t)$.

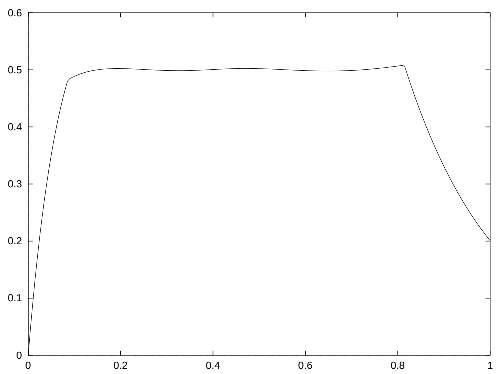

Fig. 3 Optimal trajectory of the state $x(t)$.

Each value on the chromosomes is expressed by Gray coding of 16 bits. The population size is 1,200, and the GA is performed through 250 generations. The initial values of the chromosomes are generated by uniformly random numbers. In the selection stage of the GA, an individual survives to the number of the integer part of the fitness value divided by the mean of the fitness value, i.e. if the value is greater than two, the individual is copied. Moreover, the individual that has the best value of J_1 in the group in which members satisfy $|\phi_1(x(t_1))| \leq \epsilon_1$ survives at discretion. The remaining individuals are selected randomly according to fitness values. It is clear that the best individual also survives by the algorithm of the selection. The crossover rate is 70%, and the crossover is performed at the 16 bits boundary only. Uniform crossover and one-point crossover are applied with fifty-fifty probability. The mutation rate is 20%, and in the mutation stage a locus of one bit selected randomly is flipped. The best individual and the individual that has the best J_1 value in the group in which members satisfy $|\phi_1(x(t_1))| \leq \epsilon_1$ are excluded in the crossover and mutation stages.

The optimal input obtained by the proposed method is shown in **Fig. 2**. **Figure 3** shows the optimal trajectory of x, **Fig. 4** shows the optimal trajectory of p, and **Fig. 5**

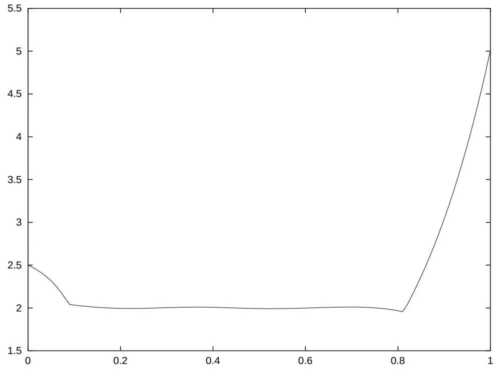

Fig. 4 Optimal trajectory of the co-state $p(t)$.

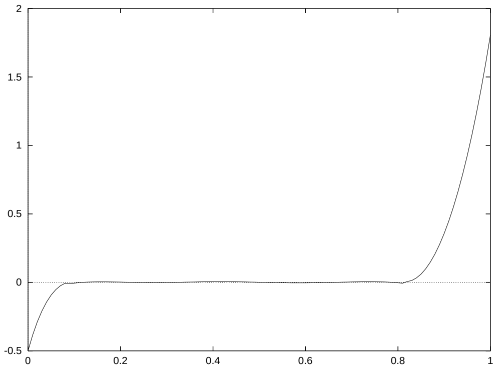

Fig. 5 Optimal coefficient $H_1(x,p)$ of the input in the Hamiltonian function.

shows the value of $H_1(x,p)$. The obtained optimal cost J is -5.26641. The terminal constraint error of the obtained solution is $\phi(x(t_1)) = 7.34 \times 10^{-4}$, which is less than 30% of the pre-specified tolerance $\epsilon_1 = 0.0025$. Due to the linearity of the input in the Hamiltonian, the period during which the input of Fig. 2 is not on the boundary of the input constraint indicates the singular control. The obtained solution shows the outline of the optimal input. Several solutions using other random seeds are tested, and all solutions are similar to Figs. 2–5. In Fig. 5, $H_1(x,p)$, the coefficient of the input in the Hamiltonian, is almost zero on the singular arc, but x and p in Figs. 3 and 4 are slightly perturbed on the singular arc. When all inputs are included linearly in the Hamiltonian, i.e., when $m_2 = 0$, $\{H_i, H_j\}$ $(i,j = 0,1,\ldots,m_1)$ also vanish on the singular arc by the generalized Legendre Clebsch condition, where $\{\cdot,\cdot\}$ denotes the Poisson bracket. Therefore, to suppress the perturbation of x and p on the singular arc, it may be effective to introduce this condition into the extended performance index J_{GA}.

In this example, parameters a_1 and k_1 that are changed adaptively oscillate with small amplitudes in the GA procedure. Since the individual that has the best J_1 value in the subpopulation in which individuals satisfy $|\phi_1(x(t_1))| \leq \epsilon_1$, survives infallibly, this oscillation causes little trouble for obtaining a 'good' solution. The oscillation is a useful phenomenon for maintaining variety in the population.

6. Conclusions

We proposed a new numerical method solving the optimal control problem with singular arcs and terminal constraints using the genetic algorithm. To handle the terminal constraints properly, Lagrange multipliers were adopted, with this information included in the chromosomes. The extended performance index using the necessary conditions of the optimality was proposed, and the weighting coefficients in the extended performance index varied in every generation of the GA. Introduction of the GA increased the calculation time. However, this kind of problem is always solved in off-line rather than real-time calculation, and the development of computers in recent years makes such a method practicable. We believe that obtaining a good-quality solution is important, even though a much time is necessary for calculation.

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