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A Study on the Convergence of the Interpolation Point Augmentaion Method for L_1 and H^∞ Control Problems†

Yoshito OHTA* and Osamu GOTO**

This paper studies the convergence of the interpolation augmentation method when applied to multi-block H^∞ and L_1 control problems. Firstly, it is shown that the norm of the Hankel operator associated to the augmented 1-block problems converges to the norm of the Hankel Toeplitz operator of the multi-block problem in the H^∞ control setting. Secondly, it is shown that the norm of the upper bound sequence approaches the optimum under some conditions for L_1 2-block problems to guarantee the suboptimality of the sequence.

Key Words: H^∞ control, Hankel-Toeplitz operator, L_1 control, weak-$*$ topology.

1. Introduction

One of the main idea of robust control is the exploitation of the induced norm of a closed loop map in the feedback system. If we measure the magnitude of the signal by its absolute maximal peak, then the induced norm turns out to be the L_1 norm of the impulse response. Hence the control design involves the minimization of the L_1 norm, and this is the motivation of the L_1 control theory [1)].

The minimization of the L_1 norm is solved in 2) for so-called 1-block discrete-time problems and in 3) for so-called 1-block continuous-time problems. However, an optimal solution for so-called multi-block problems which deal with trade-off between various control specifications is not obtained by the methods in 2), 3).

In 4) a new method to solve multi-block discrete-time ℓ_1 control problems were proposed. It is called delay augmentation method (DA method). The idea of the DA method is to augment a multi-block problem to a sequence of 1-block problems from which we derive lower and upper bound sequences for the original problem using optimal solutions of the augmented problems. The method is inherently for discrete-time problems because it explicitly uses properties of the norm space of discrete-time systems. Therefore, the DA method is not applicable to continuous-time problems.

The interpolation point augmentation method [5)] exploits properties of the weak-$*$ topology instead of those of a particular norm space, and gives an optimal solution of continuous-time L_1 control problems. Since the convergence is derived by the properties of weak-$*$ closed subspace, the augmented blocks can be constructed to have unstable zeros and need not be pure delays. Since the weak-$*$ topology is a key factor, the method can be applied to H^∞ control as well.

There are a couple of issues on the convergence of the interpolation point augmentation method. One of them is to strengthen the theoretical background of the block augmentation such as a relation to existing theories. Another issue is to give a stronger convergence than the weak-$*$ convergence of the upper bound sequence. Note that the weak-$*$ convergence does not guarantee the suboptimality, *i.e.,* the derivation of a solution whose norm is close to the optimal within a given error.

In this paper, the above two issues are resolved. For the H^∞ control problems, we give a relation of the mixed Hankel-Toeplitz operator of the original multi-block problem and the Hankel operators of augmented 1-block problems, and show that the Hankel operators are compressions of the mixed Hankel-Toeplitz operator. The increase of the number of augmentation points corresponds the enlargement of the compression spaces, and from this we can deduce the convergence of the norm of the Hankel operators to the norm of mixed Hankel-Toeplitz operator.

For the L_1 control problems, the norm of the upper bound sequences converges to the optimal if the support of delta functions of lower bound sequences is uniformly spaced [5)]. It is shown that the same conclusion holds under a milder condition than the assumption of uniformly spaced support. If this condition is satisfied, then the suboptimality of the interpolation point augmentation method is guaranteed.

* Faculty of Engineering, Osaka University, Suita, Osaka, (Graduate School of Engineering, Osaka University, Suita, Osaka)
** Corporate Research and Development Laboratories, Sumitomo Metal Industirs, Ltd., Konohana-ku, Osaka

Table 1 Classes of input-output operators.

control problem	signal space	class of input-output stable operators
discrete-time H^∞ control	$\ell_2(\mathbb{Z}_+)$	transfer functions in $H^\infty(D)$
continuous-time H^∞ control	$L^2(\mathbb{R}_+)$	transfer functions in $H^\infty(\mathbb{C}_+)$
discrete-time ℓ_1 control	$\ell_\infty(\mathbb{Z}_+)$	convolution operators in $\ell_1(\mathbb{R}_+)$
continuous-time L_1 control	$L^\infty(\mathbb{R}_+)$	convolution operators in $L_1(\mathbb{R}_+)$

2. Interpolation point augmentation method

The so-called standard control problem is transformed into the following model matching problem using the parameterization of stabilizing controllers:

$$\gamma = \inf \left\{ \|\Phi\|_{\mathrm{ind}} : \Phi = H - UQV, Q \in \mathcal{A} \right\}, \qquad (1)$$

where $\mathcal{A}$ is a class of linear time-invariant input-output stable systems defined as in **Table 1** depending on the control problems, and $H, U, V \in \mathcal{A}$ are fixed operators determined by the generalized control plant. The set of nonnegative integers is denoted as $\mathbb{Z}_+$, the set of nonnegative real numbers is denoted as $\mathbb{R}_+$, the closed unit disk in the complex plane is denoted as D, and the open right half plane in the complex plane is denoted as $\mathbb{C}_+$.

For single-input single-output systems, the input-output relation of an element in $\mathcal{A}$ is defined as follows (see also 5)). Let u and y be elements of the input and the output signal spaces, respectively, and let $\hat{u}$ and $\hat{v}$ denote the λ transform [(1)] of u and v if the system is discrete-time and the Laplace transform of u nad v if the system is continuous-time. For (continuous and discrete-time) H^∞ problems, $\hat{H} \in H^\infty(D)$ (or $\hat{H} \in H^\infty(\mathbb{C}_+)$) defines the map $\hat{y} = \hat{H}\hat{u}$. With a slight abuse of notation, we shall denote the map as $H \in \mathcal{A}$. For discrete-time ℓ_1 problems, $H \in \ell_1$ defines the convolution map

$$y(t) = \sum_{\tau=0}^{t} u(t-\tau)H(\tau).$$

The transfer function of the map is

$$\hat{H}(\lambda) = \sum_{\tau=0}^{\infty} \lambda^\tau H(\tau).$$

It is known that $\hat{H} \in H^\infty(D)$. For continuous-time L_1 problems, $H \in M(\mathbb{R}_+)$ defines the convolution map

$$y(t) = \int_{\tau\in[0,t]} u(t-\tau)dH(\tau).$$

The transfer function of the map is

$$\hat{H}(s) = \int_{\tau\in\mathbb{R}_+} \mathrm{e}^{-s\tau} dH(\tau),$$

which is in $H^\infty(\mathbb{C}_+)$. For multi-input multi-output systems, input-output operators are defined by matrices whose entries are elements of $\mathcal{A}$.

We shall assume that the transfer functions $\hat{H}$, $\hat{U}$ and $\hat{V}$ are rational in (1). If both $\hat{U}$ and $\hat{V}$ are square, the problem (1) is called 1-block. Otherwise it is called multi-block. The infimum (1) is achieved by an optimal solution if the problem is 1 block and $\hat{U}$ and $\hat{V}$ do not have zeros on the boundary, *i.e.*, the imaginary axis if the system is continuous-time, and the unit circle if the system is discrete-time. For multi-block problems, partition H, U, and V in accordance with the sizes of U and V as follows:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \qquad (2)$$
$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}.$$

Assumption 1. The block partition (2) satisfies the following conditions.

(i) The subsystem U_1 and V_1 are square, and $\det \hat{U}_1$ and $\det \hat{V}_1$ are not identically zero.

(ii) There are no unstable zeros on the boundary, *i.e.*, the imaginary axis if the system is continuous-time, and the unit circle if the system is discrete-time.

Note. The condition (i) is satisfied by column or row permutation if $\hat{U}$ is of full column rank and $\hat{V}$ is of full row rank. The condition (ii) is for the sake of simplicity.

Now, augment U and V as

$$U_a^{(k)} = \begin{bmatrix} U & X_c^{(k)} \end{bmatrix} = \begin{bmatrix} U_1 & 0 \\ U_2 & X_{2,c}^{(k)} \end{bmatrix}, \qquad (3)$$

$$V_a^{(k)} = \begin{bmatrix} V \\ X_r^{(k)} \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \\ 0 & X_{2,r}^{(k)} \end{bmatrix}, \qquad (4)$$

and consider the 1-block problem

$$\gamma_a^{(k)} = \inf \left\{ \|\Phi\| : \Phi = H - U_a^{(k)} Q_a V_a^{(k)},\ Q_a \in \mathcal{A} \right\}, \qquad (5)$$

$$Q_a = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \qquad (6)$$

Assumption 2. The operators $X_{2,c}^{(k)}$ and $X_{2,r}^{(k)}$ satisfy the following conditions.

(i) Their λ transforms (Laplace transforms) are stable rational functions.

(ii) There is no zeros on the unit circle (on the imaginary axis).

(1) If z^{-1} is replaced by λ in the z transform, it is called the λ transform. The unstable region is inside the unit circle.

(iii) In the ring of stable rational functions, $X_{2,c}^{(k)}$ is a left divisor of $X_{2,c}^{(k+1)}$ and $X_{2,r}^{(k)}$ is a right divisor of $X_{2,r}^{(k+1)}$.

(iv) The zeros of any elementary divisor have an accumulation point in the unit circle (open right half plane) as $k \to \infty$.

Note. If we choose $\hat{X}_{2,c}^{(k)}$ and $\hat{X}_{2,r}^{(k)}$ as a multiple of identity matrix by a scalar transfer function, we have a concrete construction of them which satisfies Assumption 2. Since the condition (iv) depends on the zeros of the minimum elementary divisor, it is equivalent that the blocking zeros of $\hat{X}_{2,c}^{(k)}$ and $\hat{X}_{2,r}^{(k)}$ have an accumulation point in the open right half plane.

If Assumption 2 holds, then the problem (5) has an optimal solution which we denote

$$\Phi^{(k)} = H - U_a^{(k)} Q^{(k)} V_a^{(k)}, \tag{7}$$

$$Q^{(k)} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} \\ Q_{21}^{(k)} & Q_{22}^{(k)} \end{bmatrix}. \tag{8}$$

Define

$$\Psi^{(k)} = H - U Q_{11}^{(k)} V. \tag{9}$$

Then because $\Phi^{(k)}$ is optimal for a relaxed problem and $\Psi^{(k)}$ is feasible for the primal problem, it follows that

$$\left\| \Phi^{(k)} \right\| = \gamma_a^{(k)} \le \gamma \le \left\| \Psi^{(k)} \right\|.$$

Furthermore, the lower bound sequence $\left\{ \Phi^{(k)} \right\}$ and the upper bound sequence $\left\{ \Psi^{(k)} \right\}$ approaches an optimal solution of the multi-block model matching problem (1) in the following sense.

Theorem 1. Suppose that the augmented 1-block problem (5) satisfies Assumption 2. Then a sequence of optimal solutions (7) have a weak-$*$ convergent sequence $\left\{ \Psi^{(k_i)} \right\}$. Furthermore, it follows that

$$\text{weak-}*\lim \Phi^{(k_i)} = \text{weak-}*\lim \Psi^{(k_i)} =: \Phi^{(o)},$$

and $\Phi^{(o)}$ is optimal for (1).

Proof. From (7) and (9),

$$\Phi^{(k)} = \Psi^{(k)} - \left(R_1^{(k)} + R_2^{(k)} + R_3^{(k)} \right), \tag{10}$$

where

$$R_1^{(k)} = X_c^{(k)} Q_{21}^{(k)} V,$$
$$R_2^{(k)} = U Q_{12}^{(k)} X_r^{(k)},$$
$$R_3^{(k)} = X_c^{(k)} Q_{22}^{(k)} X_r^{(k)}.$$

From (10),

$$\Phi_{11}^{(k)} = \Psi_{11}^{(k)},$$
$$\Phi_{12}^{(k)} = \Psi_{12}^{(k)} - U_1 Q_{12}^{(k)} X_{2,r}^{(k)},$$
$$\Phi_{21}^{(k)} = \Psi_{21}^{(k)} - X_{2,c}^{(k)} Q_{21}^{(k)} V_1,$$
$$\Phi_{22}^{(k)} = \Psi_{22}^{(k)} - \Big(U_2 Q_{12}^{(k)} X_{2,r}^{(k)} + X_{2,c}^{(k)} Q_{21}^{(k)} V_2 + X_{2,c}^{(k)} Q_{22}^{(k)} X_{2,r}^{(k)} \Big).$$

Applying Lemma 5 in 5), we see that the boundedness of $\left\{ \Phi_{11}^{(k)} \right\}$ implies the boundedness of $\left\{ Q_{11}^{(k)} \right\}$ and hence the boundedness of $\{\Psi(k)\}$. This in turn implies the boundedness of $\left\{ U_1 Q_{12}^{(k)} X_{2,r}^{(k)} \right\}$ and $\left\{ X_{2,c}^{(k)} Q_{21}^{(k)} V_1 \right\}$. Applying again Lemma 5 in 5), we conclude that $\left\{ Q_{12}^{(k)} X_{2,r}^{(k)} \right\}$, $\left\{ X_{2,c}^{(k)} Q_{21}^{(k)} \right\}$ and $\left\{ X_{2,c}^{(k)} Q_{22}^{(k)} X_{2,r}^{(k)} \right\}$ are bounded. Applying a modified version of Lemma 4 in 5) for multi-input multi-output systems, we see that $Q_{12}^{(k)} X_{2,r}^{(k)} \to 0$, $X_{2,c}^{(k)} Q_{21}^{(k)} \to 0$ and $X_{2,c}^{(k)} Q_{22}^{(k)} X_{2,r}^{(k)} \to 0$ in the weak-$*$ topology. □

3. Interpolation point augmentation and mixed Hankel-Toeplitz operator

It is well known that the optimal cost of a 1-block H^∞ control problem is equal to the norm of a Hankel operator, and that the optimal cost of a multi-block H^∞ control problem is equal to the norm of a mixed Hankel-Toeplitz operator [6]. In this section, we shall investigate the relation between the Hankel operators corresponding to augmented 1-block problems and the mixed Hankel-Toeplitz operator corresponding to the original multi-block problem. Specifically, it is shown that the norm of the Hankel operators converges to the norm of the mixed Hankel-Toeplitz operator. This is an alternate proof of the convergence of the lower bounds to the optimal cost. For the sake of simplicity, we shall treat the continuous-time systems in this section though the theory applies both continuous and discrete-time systems.

3.1 Mixed Hankel-Toeplitz operator

Note that the Fourier transform is a unitary map between $L^2(\mathbb{R})$ of the time domain signals and $L^2(j\mathbb{R})$ of the frequency domain signals, *i.e.*, for $f \in L^2(\mathbb{R})$ and its Fourier transform $\hat{f} \in L^2(j\mathbb{R})$, there holds

$$\|f\|^2 = \int_{-\infty}^{\infty} f(t)^* f(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(j\omega)^* \hat{f}(j\omega) d\omega = \left\| \hat{f} \right\|^2.$$

Let partition the time axis $\mathbb{R}$ into the positive time interval $\mathbb{R}_+$ and the negative time interval $\mathbb{R}_-$. This extends to the frequency domain partition

$$L^2(j\mathbb{R}) = H^2(\mathbb{C}_+) \oplus H^2(\mathbb{C}_-),$$

where $\mathbb{C}_-$ is the left open half plane of the complex plane. Let Π_+ and Π_- be the orthogonal projection form $L^2(j\mathbb{R})$ onto $H^2(\mathbb{C}_+)$ and $H^2(\mathbb{C}_-)$, respectively.

Let $L^\infty(j\mathbb{R})$ be the matrix valued bounded functions on the imaginary axis. By the multiplication $\hat{M}(j\omega)\hat{f}(j\omega)$ for $\hat{M} \in L^\infty(j\mathbb{R})$ and $\hat{f} \in L^2(j\mathbb{R})$, we can define a map from $L^2(j\mathbb{R})$ to $L^2(j\mathbb{R})$, which we shall denote $M : L^2(j\mathbb{R}) \to L^2(j\mathbb{R})$, and call $\hat{M}$ as the symbol of the map M. Taking non-tangential limit to the imaginary axis, we can identify $H^\infty(\mathbb{C}_+)$ as a subspace of $L^\infty(j\mathbb{R})$. The map whose symbol is in $H^\infty(\mathbb{C}_+)$ leaves $H^2(\mathbb{C}_+)$ invariant.

Consider the H^∞ model matching problem (1). We can assume that $\hat{U}$ is inner because the outer part can be included in the free parameter $\hat{Q}$. By the same reason, we assume that $\hat{V}$ is co-inner. For a stale rational function $\hat{M}(s)$, the para-conjugate $\hat{M}^\sim$ is defined as $\hat{M}^\sim(s) = \hat{M}(-s)^{\mathrm{T}}$.

If the problem (1) is 1-block, then the optimal value is equal to the Hankel operator

$$\Pi_- U^\sim H V^\sim|_{H^2(\mathbb{C}_+)}\,.$$

If the problem (1) is multi-block, then define square inner functions

$$\begin{bmatrix}\hat{U} & \hat{U}_c\end{bmatrix}, \quad \begin{bmatrix}\hat{V} \\ \hat{V}_r\end{bmatrix},$$

by inserting $\hat{U}_c$ and $\hat{V}_r$ [7]. The the optimal value of the problem (1) is equal to the norm of the mixed Hankel-Toeplitz operator [6]

$$\Gamma_{HT} = \begin{bmatrix}\Pi_- U^\sim H V^\sim|_{H^2(\mathbb{C}_+)} & \Pi_- U^\sim H V_r^\sim \\ U_c^\sim H V^\sim|_{H^2(\mathbb{C}_+)} & U_c^\sim H V_r^\sim.\end{bmatrix} \qquad (11)$$

3.2 Relation to interpolation augmentation method

The augmented blocks in (3) and (4) satisfy the following in addition to Assumption 2.

Assumption 3. The augmented blocks $U_a^{(k)}$ and $V_a^{(k)}$ satisfy the following conditions.

(i) The transfer functions $\hat{U}^\sim \hat{X}_c^{(k)}$ and $\hat{X}_r^{(k)}\hat{V}^\sim$ are stable rational functions.

(ii) The determinants of $\hat{U}^\sim \hat{X}_c^{(k)}$ and $\hat{X}_r^{(k)}\hat{V}^\sim$ are not identically zero, and do not have a zero on the imaginary axis including the infinity.

Note. A construction of $\hat{X}_c^{(k)}$ and $\hat{X}_r^{(k)}$ to satisfy Assumption 3 is done as follows. The condition (i) is satisfied if we cancel the unstable poles of $\hat{U}^\sim$ and $\hat{V}^\sim$ by placing zeros to $\hat{X}_c^{(k)}$ and $\hat{X}_r^{(k)}$. Because the $(1,1)$ block $\hat{U}_1$ of $\begin{bmatrix}\hat{U} & \hat{U}_c\end{bmatrix}$ has no zeros on the boundary, the $(2,2)$ block of $\begin{bmatrix}\hat{U} & \hat{U}_c\end{bmatrix}^\sim$ is nonsingular on the boundary, and hence the condition (ii) is satisfied.

The Hankel operator corresponding the augmented 1-block problem (5) is computed as follows. Since

$$\begin{bmatrix}\hat{U}^\sim \\ \hat{U}_c^\sim\end{bmatrix}\hat{U}_a^{(k)} = \begin{bmatrix}I & \hat{U}^\sim \hat{X}_c^{(k)} \\ 0 & \hat{U}_c^\sim \hat{X}_c^{(k)}\end{bmatrix},$$

the inner-outer factorization of (3) is

$$\hat{U}_a^{(k)} = \begin{bmatrix}\hat{U} & \hat{U}_c \hat{Y}_{c,i}^{(k)}\end{bmatrix}\begin{bmatrix}I & \hat{U}^\sim \hat{X}_c^{(k)} \\ 0 & \hat{Y}_{c,o}^{(k)}\end{bmatrix},$$

where $\hat{U}_c^\sim \hat{X}_c^{(k)} = \hat{Y}_{c,i}^{(k)}\hat{Y}_{c,o}^{(k)}$ is the inner-outer factorization. Similarly, (4) is factorized as

$$\hat{V}_a^{(k)} = \begin{bmatrix}I & 0 \\ \hat{X}_r^{(k)}\hat{V}^\sim & \hat{Y}_{r,co}^{(k)}\end{bmatrix}\begin{bmatrix}\hat{V} \\ \hat{Y}_{r,ci}^{(k)}\hat{V}_r\end{bmatrix},$$

where $\hat{X}_r^{(k)}\hat{V}_r^\sim = \hat{Y}_{r,co}^{(k)}\hat{Y}_{r,ci}^{(k)}$ is the coouter-coinner factorization. Define

$$\Gamma_H^{(k)} = \begin{bmatrix}\Pi_- U^\sim H V^\sim|_{H^2(\mathbb{C}_+)} & \Pi_- U^\sim H V_r^\sim Y_{r,ci}^{(k)\sim}\Big|_{H^2(\mathbb{C}_+)} \\ \Pi_- Y_{c,i}^{(k)\sim} U_c^\sim H V^\sim\Big|_{H^2(\mathbb{C}_+)} & \Pi_- Y_{c,i}^{(k)\sim} U_c^\sim H V_r^\sim Y_{r,ci}^{(k)\sim}\Big|_{H^2(\mathbb{C}_+)}\end{bmatrix}. \qquad (12)$$

Then it turns out that $\Gamma_H^{(k)}$ is the Hankel operator corresponding to the augmented 1-block problem (5).

Theorem 2. Assume that Assumptions 2 and 3 hold. Then the Hankel operator (12) and the mixed Hankel-Toeplitz operator (11) satisfy

$$\lim_{k\to\infty}\left\|\Gamma_H^{(k)}\right\| = \|\Gamma_{HT}\|\,.$$

Proof. The operators (12) and (11) are related by the equation

$$\Gamma_H^{(k)} = \begin{bmatrix}I & 0 \\ 0 & \Pi_- Y_{c,i}^{(k)\sim}\end{bmatrix}\Gamma_{HT}\begin{bmatrix}I & 0 \\ 0 & Y_{r,ci}^{(k)\sim}\Big|_{H^2(\mathbb{C}_+)}\end{bmatrix}.$$

Define

$$\Lambda^{(k)} = \begin{bmatrix}I & 0 \\ 0 & Y_{c,i}^{(k)}\Big|_{H^2(\mathbb{C}_-)}\end{bmatrix}\Gamma_H^{(k)}\begin{bmatrix}I & 0 \\ 0 & \Pi_+ Y_{r,ci}^{(k)}\end{bmatrix}.$$

Then it follows that $\Lambda_{(k)} \to \Gamma_{HT}$ (SOT) as $k \to \infty$, where (SOT) means that the convergence is in the strong operator topology [8]. Indeed, let

$$\Pi_c^{(k)} = \begin{bmatrix}I & 0 \\ 0 & Y_{c,i}^{(k)}\Big|_{H^2(\mathbb{C}_-)}\Pi_- Y_{c,i}^{(k)\sim}\end{bmatrix},$$

$$\Pi_r^{(k)} = \begin{bmatrix}I & 0 \\ 0 & Y_{r,ci}^{(k)\sim}\Big|_{H^2(\mathbb{C}_+)}\Pi_+ Y_{r,ci}^{(k)}\end{bmatrix}.$$

Then $\Pi_c^{(k)*} = \Pi_c^{(k)}$, $\Pi_r^{(k)*} = \Pi_r^{(k)}$, $\Pi_c^{(k)2} = \Pi_c^{(k)}$ and $\Pi_r^{(k)2} = \Pi_r^{(k)}$ imply that these operators are orthogonal projections. Furthermore, $\Pi_c^{(k)} \to I$ (SOT). Note that $\cdots \supset Y_{c,i}^{(k)} H^2(\mathbb{C}_+) \supset Y_{c,i}^{(k+1)} H^2(\mathbb{C}_+) \supset \cdots$, and $\cap_k Y_{c,i}^{(k)} H^2(\mathbb{C}_+) = (0)$ where (0) is the zero subspace. Hence considering the orthogonal complements in $L^2(j\mathbb{R})$ we have $\cdots \subset Y_{c,i}^{(k)} H^2(\mathbb{C}_-) \subset Y_{c,i}^{(k+1)} H^2(\mathbb{C}_-) \subset \cdots$, and $\oplus_k Y_{c,i}^{(k)} H^2(\mathbb{C}_-) = L^2(j\mathbb{R})$. This implies that given $\epsilon > 0$ and $f \in L^2(j\mathbb{R})$ there is $k > 0$ and $g \in H^2(\mathbb{C}_-)$ such that $\left\| f - Y_{c,i}^{(k)} g \right\| < \epsilon$. Then

$$\begin{aligned}
&\left\| f - Y_{c,i}^{(k)} \Big|_{H^2(\underline{C})} \Pi_- Y_{c,i}^{(k)\sim} f \right\| \\
&= \left\| f - Y_{c,i}^{(k)} \Big|_{H^2(\underline{C})} \Pi_- Y_{c,i}^{(k)\sim} \left(f - Y_{c,i}^{(k)} g \right) \right. \\
&\qquad \left. - Y_{c,i}^{(k)} \Big|_{H^2(\underline{C})} \Pi_- Y_{c,i}^{(k)\sim} Y_{c,i}^{(k)} g \right\| \\
&\le \left(1 + \left\| Y_{c,i}^{(k)} \Big|_{H^2(\underline{C})} \Pi_- Y_{c,i}^{(k)\sim} \right\| \right) \left\| f - Y_{c,i}^{(k)} g \right\| \\
&\le 2\epsilon,
\end{aligned}$$

which shows that $\Pi_c^{(k)} \to I$ (SOT). We can show $\Pi_r^{(k)} \to I$ (SOT) similarly. These in turn imply that $\Lambda^{(k)} = \Pi_c^{(k)} \Gamma_{HT} \Pi_r^{(k)} \to \Gamma_{HT}$ (SOT) [8]. From this we have $\liminf \left\| \Lambda^{(k)} \right\| \ge \| \Gamma_{HT} \|$. On the other hand, $\left\| \Lambda^{(k)} \right\| = \left\| \Pi_c^{(k)} \Gamma_{HT} \Pi_r^{(k)} \right\| \le \| \Gamma_{HT} \|$. Hence $\lim_k \left\| \Lambda^{(k)} \right\| = \| \Gamma_{HT} \|$. Since $\left\| \Lambda^{(k)} \right\| = \left\| \Gamma_H^{(k)} \right\|$, this proves the theorem. □

Note. A sequence $\{T_k\}$ of operators in a Hilbert space X converges to T in the strong operator topology (SOT) if for any $x \in X$ there holds $\|T_k x - T x\| \to 0$. See 8) for detail.

Note. The Hankel operator $\Gamma_H^{(k)}$ and the operator $\Lambda^{(k)}$ in the proof of Theorem 2 are compressions of the mixed Hankel-Toeplitz operator Γ_{HT}. The augmented 1-block problems approached the original multi-block problem in the sense the orthogonal projections $\Pi_c^{(k)}$ and $\Pi_r^{(k)}$ converge to the identity operators in the strong operator topology.

4. Application to L_1 control

The interpolation point augmentation method yields an upper bound sequence that converges to an optimal solution of the multi-block problem in the weak-∗ topology. If a sequence $\left\{ \Psi^{(k)} \right\}$ converges to $\Phi^{(o)}$ in weak-∗ topology,

$$\left\| \Phi^{(o)} \right\| \le \liminf_k \left\| \Psi^{(k)} \right\|.$$

This shows that if the number of the augmentation is finite then we cannot guarantee that $\left\| \Psi^{(k)} \right\|$ is close to the optimal value.

In 5), the norm convergence of the upper bound sequence was proved for a class of column 2-block problems under the assumption that the solutions of augmented 1-block problems have uniform space for delta functions. In this paper, we shall show the norm convergence with less restricted assumption. Notice that the conclusion of 5) is wrong unless the same assumption on the direct term of $\hat{U}_2 \hat{U}_1^{-1}$.

Consider the column 2-block problem

$$\gamma = \inf \left\{ \|\Phi\| : \Phi = H - UQ, \quad Q \in M(\mathbb{R}_+) \right\}, \tag{13}$$

where

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}.$$

The augmented 1-block problem is

$$\gamma_a^{(k)} = \inf \left\{ \|\Phi\| : \Phi = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} - \begin{bmatrix} U_1 & 0 \\ U_2 & X_{2,c}^{(k)} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \right.$$
$$\left. Q_1, Q_2 \in M(\mathbb{R}_+) \right\}.$$

Lower and upper bound sequences are

$$\Phi^{(k)} = \begin{bmatrix} \Phi_1^{(k)} \\ \Phi_2^{(k)} \end{bmatrix} = \begin{bmatrix} H_1 - U_1 Q_1^{(k)} \\ H_2 - U_2 Q_1^{(k)} - X_{2,c} Q_2^{(k)} \end{bmatrix},$$

and

$$\Psi^{(k)} = \begin{bmatrix} \Psi_1^{(k)} \\ \Psi_2^{(k)} \end{bmatrix} = \begin{bmatrix} H_1 - U_1 Q_1^{(k)} \\ H_2 - U_2 Q_1^{(k)} \end{bmatrix}.$$

Notice that

$$\begin{aligned}
\Psi_1^{(k)} &= \Phi_1^{(k)}, \\
\Psi_2^{(k)} &= \Phi_2^{(k)} + X_{2,c}^{(k)} Q_2^{(k)}
\end{aligned}$$

hold. Let $\Phi_{1,i}^{(o)}$ be the i-th row of $\Phi_1^{(o)}$. If a matrix consists of rows having at most one nonzero element, then it is called a row selection matrix.

Theorem 3. Consider the column 2-block problem (13). Suppose that $\hat{H}(s)$ is strictly proper and the direct term of $\hat{U}_2 \hat{U}_1^{-1}$ is a row selection matrix. Furthermore, suppose $\Phi^{(o)}$ has no absolute continuous part, and $\left\| \Phi_1^{(o)} \right\| = \left\| \Phi^{(o)} \right\|$. Then it follows that

$$\lim_{k \to \infty} \left\| \Psi^{(k)} \right\| = \left\| \Phi^{(o)} \right\|.$$

Proof. First notice that

$$\begin{aligned}
\hat{\Psi}_2^{(k)} &= \hat{H}_2 - \hat{U}_2 \hat{U}_1^{-1} \hat{H}_1 + \hat{U}_2 \hat{U}_1^{-1} \hat{\Phi}_1^{(k)}, \\
\hat{\Psi}_2^{(o)} &= \hat{H}_2 - \hat{U}_2 \hat{U}_1^{-1} \hat{H}_1 + \hat{U}_2 \hat{U}_1^{-1} \hat{\Phi}_1^{(o)}.
\end{aligned}$$

Since $\hat{U}_1^{-1}$ is rational, two-sided inverse Laplace transform is well defined, and we have

$$\begin{aligned}
\Psi_2^{(k)} &= W_1 + W_{20} \Phi_1^{(k)} + W_{2M} * \Phi_1^{(k)}, \\
\Psi_2^{(o)} &= W_1 + W_{20} \Phi_1^{(o)} + W_{2M} * \Phi_1^{(o)},
\end{aligned}$$

in $L_1(\mathbb{R}_+) \cap M(\mathbb{R}_+)$, where $W_1 \in L_1(\mathbb{R})$ is the inverse Laplace transform of $\hat{H}_2 - \hat{U}_2 \hat{U}_1^{-1} \hat{H}_1$, W_{20} is the direct term of $\hat{U}_2 \hat{U}_1^{-1}$, and $W_{2M} \in L_1(\mathbb{R})$ is the inverse Laplace transform of the strictly proper part of $\hat{U}_2 \hat{U}_1^{-1}$. Notice that $W_{2M} \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$. Then Lemma 1 in Appendix shows that $W_{2M} * \Phi^{(k)} \to W_{2M} * \Phi^{(o)}$. Hence it follows that

$$\left\| \Psi_2^{(k)} \right\| \to \left\| \Psi_2^{(o)} \right\| = \left\| \Phi_2^{(o)} \right\| \leq \left\| \Phi^{(o)} \right\|.$$

□

5. Conclusion

In this paper, we considered the convergence of the lower and upper bound sequences of the interpolation point augmentation method. For the lower bound sequence, the norms of the Hankel operators corresponding to the augmented 1-block problems approaches the norm of the mixed Hankel-Toeplitz operator of the multi-block problem. This links the interpolation point augmentation method and the existing H^∞ control theory. For the upper bound sequence, the norm of the upper bound sequence converges to the optimal value under some conditions for continuous-time L_1 control problems. This convergence is stronger than the weak-$*$ convergence.

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Appendix A. Functional spaces related to continuous L_1 control

Notice that $M(\mathbb{R}_+)$ is a Banach algebra when the multiplication is defined by the convolution. We shall extend the time axis to include the negative time and consider the space $L_1(\mathbb{R})$ and $C_0(\mathbb{R})$. If $G \in C_0(\mathbb{R})$ and $\Phi \in M(\mathbb{R}_+)$, then

$$\langle G, \Phi \rangle = \int_{\tau \in \mathbb{R}_+} G(\tau) d\Phi(\tau),$$

from which we regard $M(\mathbb{R}_+)$ as a subspace of the dual space of $C_0(\mathbb{R})$. If $G \in L_1(\mathbb{R})$ and $\Phi \in M(\mathbb{R}_+)$, then we shall define $F = G * \Phi \in L_1(\mathbb{R})$ as

$$F(t) = \int_{\tau \in \mathbb{R}_+} G(t-\tau) d\Phi(\tau). \qquad \text{(A.1)}$$

Notice that $F \in L_1(\mathbb{R})$ is concluded by the similar argument as in 9).

Let $X \subset \mathbb{R}$ be a Borel set. The characteristic function χ is defined as

$$\chi(t) = \begin{cases} 1, & (t \in X) \\ 0, & (t \notin X) \end{cases}.$$

With a alight abuse of notation, for $\Phi \in M(\mathbb{R}_+)$, we define $\chi\Phi \in M(\mathbb{R}_+)$ as

$$\int_{\tau \in T} d\chi\Phi = \int_{\tau \in T \cap X} d\Phi,$$

where $T \subset \mathbb{R}$ is a measurable set. For $G \in L_1(\mathbb{R})$, define $\chi G \in L_1(\mathbb{R})$ as

$$(\chi G)(t) = \chi(t) G(t).$$

In this section, we shall see some important properties of these spaces.

Property 1. Let $\Phi \in M(\mathbb{R}_+)$ and $\epsilon > 0$. Then for sufficiently large t_1, $\|\chi\Phi\| < \epsilon$, where χ is the characteristic function of the interval $[t_1, \infty)$.

Proof. This follows from the fact that the total variation of Φ is bounded. □

Property 2. Let $\left\{\Phi^{(k)}\right\} \subset M(\mathbb{R}_+)$ be a bounded weak-$*$ convergent sequence. Let $\Phi^{(o)} = \text{weak-}*\lim \Phi^{(k)}$. Suppose that $\lim \|\Phi\ (k)\| = \left\|\Phi^{(o)}\right\|$ holds. Then for any $\epsilon > 0$, there is t_1 such that $\left\|\chi\Phi^{(o)}\right\| < \epsilon$ and $\left\|\chi\Phi^{(k)}\right\| < \epsilon$ for sufficiently large k, where χ is the characteristic function of the interval $[t_1, \infty)$.

Proof. A singular measure is supported on a set whose (Lebesgue) measure is zero. Hence the measure of the union of the support sets of the singular part of $\Phi^{(k)}$, $k = 1, 2, \cdots$, and $\Phi^{(o)}$ is also zero. Choose $0 < t_0 < t_1$ in such a way that t_0 and t_1 are not on the support set of the singular parts and $\left\|\chi_0\Phi^{(o)}\right\| < \epsilon/3$, where χ_0 is the characteristic function of the interval $[t_0, \infty)$ (see Property 1). Choose $F \in C_0(\mathbb{R}_+)$ in such a way that $\|F\| = 1$, $F(t) = 0$ if $t > t_1$, and $\left\langle F, \Phi^{(o)} \right\rangle > \left\|\Phi^{(o)}\right\| - \epsilon/2$. If there us no k such that $\left\|\chi\Phi^{(k)}\right\| < \epsilon$, then we can choose a sequence of integers $\{k_i\}$. $k_i \to \infty$ such that $\left\langle F, \Phi^{(k_i)} \right\rangle < \left\|\Phi^{(k_i)}\right\| - \epsilon$. But this implies that

$\left\langle F, \Phi^{(o)} \right\rangle = \lim \left\langle F, \Phi^{(k_i)} \right\rangle \le \left\| \Phi^{(o)} \right\| - \epsilon$, which is a contradiction. □

Property 3. If $\Phi \in M(\mathbb{R}_+)$ and $G \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$, then $G * \Phi$ is a continuous function.

Proof. Since $G \in C_0(\mathbb{R})$, G is uniformly continuous, or for any $\epsilon > 0$, there is $\delta > 0$ such that $|h| < \delta$ implies $|G(t+h) - G(t)| < \epsilon$. Hence

$$\begin{aligned}
&|F(t+h) - F(t)| \\
&= \left| \int_{\tau \in \mathbb{R}_+} \left(G(t+h-\tau) - G(t-\tau)\right) d\Phi(\tau) \right| \\
&\le \int_{\tau \in \mathbb{R}_+} |G(t+h-\tau) - G(t-\tau)|\, d\,|\Phi|\,(\tau) \\
&< \epsilon \,\|\Phi\|.
\end{aligned}$$

□

Property 4. Let $\Phi \in M(\mathbb{R}_+)$ and $G \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$. For any $\epsilon > 0$, there is $t_0 < 0$ such that

(i) $\sup |\chi_0 \left(G * \Phi\right)(t)| < \epsilon$,

(ii) $\|\chi_0 \left(G * \Phi\right)\|_1 < \epsilon$,

where χ_0 is the characteristic function of the interval $(-\infty, t_0]$.

Proof. (Uniform norm): If t_0 is sufficiently small, since $G \in C_0(\mathbb{R})$ $\sup |\chi_0 G(t)| < \epsilon / \|\Phi\|$. Hence if $t < t_0$

$$\begin{aligned}
|\left(G * \Phi\right)(t)| &= \left| \int_{\tau \in \mathbb{R}_+} G(t-\tau) d\Phi(\tau) \right| \\
&\le \int_{\tau \in \mathbb{R}_+} |G(t-\tau)|\, d\,|\Phi|\,(\tau) < \epsilon.
\end{aligned}$$

(1 norm): If t_0 is sufficiently small, since $G \in L_1(\mathbb{R})$ $\|\chi_0 G\|_1 < \epsilon / \|\Phi\|$. Then

$$\begin{aligned}
\|\chi_0 \left(G * \Phi\right)\|_1 &= \int_{-\infty}^{t_0} \left| \int_{\tau \in \mathbb{R}_+} G(t-\tau) d\Phi(\tau) \right| dt \\
&\le \int_{-\infty}^{t_0} \int_{\tau \in \mathbb{R}_+} |G(t-\tau)|\, d\,|\Phi|\,(\tau) dt \\
&= \int_{\tau \in \mathbb{R}_+} \int_{-\infty}^{t_0 - \tau} |G(t)|\, dt d\,|\Phi|\,(\tau) < \epsilon.
\end{aligned}$$

□

Property 5. Let $G \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$. Let $\left\{\Phi^{(k)}\right\} \subset M(\mathbb{R}_+)$ be a bounded sequence. Suppose for any $\epsilon > 0$, there is $t_1 > 0$ such that $2 \max\left\{\|G\|_\infty, \|G\|_1\right\} \left\| \chi_1 \Phi^{(k)} \right\| < \epsilon$, where χ_1 is the characteristic function of the interval $[t_1, \infty)$. Then for sufficiently large $t_2 > t_1$,

(i) $\sup \left| \chi_2 \left(G * \Phi^{(k)}\right)(t) \right| < \epsilon$,

(ii) $\left\| \chi_2 \left(G * \Phi^{(k)}\right) \right\|_1 < \epsilon$,

where χ_2 is the characteristic function of the interval $[t_2, \infty)$.

Proof. If $G = 0$, then the conclusions are trivial. Hence we shall assume $G \neq 0$ henceforth. (Uniform norm): Since $G \in C_0(\mathbb{R})$, choose $t_2 > t_1$ in such a way that $t > t_2 - t_1$ implies $|G(t)| < \epsilon / \left(2 \sup_k \left\|\Phi^{(k)}\right\|\right)$. Let χ_0 be the characteristic function of the interval $[0, t_1)$. Then $t > t_2$ implies

$$\begin{aligned}
&\left| \left(G * \Phi^{(k)}\right)(t) \right| \\
&= \left| \int_{\tau \in \mathbb{R}_+} G(t-\tau) d\Phi^{(k)}(\tau) \right| \\
&\le \int_{\tau \in \mathbb{R}_+} |G(t-\tau)|\, d\left|\chi_0 \Phi^{(k)}\right|(\tau) \\
&\qquad + \int_{\tau \in \mathbb{R}_+} |G(t-\tau)|\, d\left|\chi_1 \Phi^{(k)}\right|(\tau) \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

(1 norm): Since $G \in L_1(\mathbb{R})$, choose $t_2 > t_1$ so that $\|\chi_3 G\|_1 < \epsilon / \left(2 \sup_k \left\|\Phi^{(k)}\right\|\right)$ holds, where χ_3 is the characteristic function of the interval $[t_2 - t_1, \infty)$. Then

$$\begin{aligned}
&\left\| \chi_2 \left(G * \Phi^{(k)}\right) \right\|_1 \\
&= \int_{t_2}^{\infty} \left| \int_{\tau \in \mathbb{R}_+} G(t-\tau) d\Phi^{(k)}(\tau) \right| dt \\
&\le \int_{t_2}^{\infty} \int_{\tau \in \mathbb{R}_+} |G(t-\tau) d| \left|\Phi^{(k)}\right|(\tau) dt \\
&= \int_{\tau \in [0, t_1)} \int_{t_2 - \tau}^{\infty} |G(t)|\, dt d\left|\chi_0 \Phi^{(k)}\right|(\tau) \\
&\qquad + \int_{\tau \in [t_1, \infty)} \int_{t_2 - \tau}^{\infty} |G(t)|\, dt d\left|\chi_1 \Phi^{(k)}\right|(\tau) \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

Note. Properties 4 and 5 imply that $G * \Phi \in C_0(\mathbb{R})$ in Property 3.

Properties 1–5 are exploited to establish the following lemma which is crucial in proving Theorem 3.

Lemma 1. Let $G \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$. Let $\left\{\Phi^{(k)}\right\} \subset M(\mathbb{R}_+)$ be a bounded weak-$*$ convergent sequence, and $\Phi^o = \text{weak-}*\lim \Phi^{(k)}$. Suppose that $\lim \left\|\Phi^{(k)}\right\| = \left\|\Phi^{(o)}\right\|$ holds. Then $G * \Phi^{(k)} \to G * \Phi^{(o)}$ in $L_1(\mathbb{R})$.

Proof. Let $G_t(\bullet) = G(t - \bullet)$. Since $G \in C_0(\mathbb{R})$, $G_t \in C_0(\mathbb{R}_+)$. From the definition of convolution (A. 1), the evaluations of $G * \Phi^{(k)}$ and $G * \Phi^{(o)}$ at time t are the values of the linear functionals $\left\langle G_t, \Phi^{(k)} \right\rangle$ and $\left\langle G_t, \Phi^{(o)} \right\rangle$, respectively. Since Φ^o is the weak-$*$ limit, it follows that $\left(G * \Phi^{(k)}\right)(t) \to \left(G * \Phi^{(o)}\right)(t)$. Let $[t_a, t_b]$ be a compact interval. Let $f^{(k)}$ and $f^{(o)}$ be the restrictions of $G * \Phi^{(k)}$ and $G * \Phi^{(o)}$ on the interval, respectively. Note that Property 3 implies that $f^{(k)}$ and $f^{(o)}$ are continuous. We shall show that $f^{(k)} \to f^{(o)}$ uniformly on the interval. Fix

$t_0 \in [t_a, t_b]$ and $\epsilon > 0$. Choose $\delta^{(o)}(t_0) > 0$ in such a way that $|t - t_0| < \delta^{(o)}(t_0)$ implies $\left|f^{(o)}(t) - f^{(o)}(t_0)\right| < \epsilon$. Define

$$\delta^{(k)} = \min\left\{\delta^{(o)}(t_0), \sup\left\{\delta : |t - t_0| < \delta \text{ implies } \left|f^{(k)}(t) - f^{(k)}(t_0)\right| < \epsilon\right\}\right\}.$$

Notice that $\delta^{(k)} \to \delta^{(o)}(t_0)$. Otherwise, by choosing a subsequence of k if necessary we can select a positive number $\delta' < \delta^{(o)}(t_0)$ and $t^{(k)}$, $\left|t^{(k)} - t_0\right| < \delta'$ such that $\left|f^{(k)}(t^{(k)}) - f^{(k)}(t_0)\right| \geq \epsilon$. Since the set $\{t : |t - t_0| \leq \delta'\}$ is compact, the sequence $\left\{t^{(k)}\right\}$ has an accumulation point t_c in the set. Then it follows that $\left|f^{(o)}(t_c) - f\ (o)(t_0)\right| \geq \epsilon$, which contradicts the definition of $\delta^{(o)}(t_0)$. Consider an open cover $\{I(t_0) : t_a \leq t_0 \leq t_b\}$ of the compact interval $[t_a, t_b]$, where $I(t_0) = \left\{t : |t - t_0| < \delta^{(o)}(t_0)/2\right\}$. Choose a finite subcover, and let $\{t_{0,1}, t_{0,2}, \cdots, t_{0,N}\}$ be its index set. Choose an integer $k(t_0) > 0$ in such a way that $k > k(t_0)$ implies $\delta^{(k)} > \delta^{(o)}(t_0)/2$ and $\left|f\ (k)(t_0) - f^{(o)}(t_0)\right| < \epsilon$. Let $\delta = \min\left\{\delta^{(o)}(t_{0,i}) : 1 \leq i \leq N\right\}$, and $k_0 = \max\{k(t_{0,i}) : 1 \leq i \leq N\}$. If $k > k_0$, then for any $t \in [t_a, t_b]$

$$\begin{aligned} &\left|f^{(k)}(t) - f^{(o)}(t)\right| \\ &\leq \left|f^{(k)}(t) - f^{(k)}(t_{0,i})\right| + \left|f^{(k)}(t_{0,i}) - f^{(o)}(t_{0,i})\right| \\ &\qquad + \left|f^{(o)}(t_{0,i}) - f^{(o)}(t)\right| \\ &< 3\epsilon, \end{aligned}$$

for appropriate $t_{0,i}$. This proves the uniform convergence. Let χ_0, χ_1 and χ_2 be the characteristic functions of the intervals $(-\infty, t_0)$, $[t_0, t_2)$ and $[t_2, \infty)$, respectively, where $t_2 > t_1$. Then

$$\begin{aligned} &\left\|G * \Phi^{(k)} - G * \Phi^{(o)}\right\|_1 \\ &\leq \left\|\chi_1 G * \Phi^{(k)} - \chi_1 G * \Phi^{(o)}\right\|_1 \\ &\qquad + \left\|\chi_0 G * \Phi^{(k)}\right\|_1 + \left\|\chi_0 G * \Phi^{(o)}\right\|_1 \\ &\qquad + \left\|\chi_2 G * \Phi^{(k)}\right\|_1 + \left\|\chi_2 G * \Phi^{(o)}\right\|_1. \end{aligned}$$

The second and third terms of the right hand side can be arbitrary small for sufficiently small t_0 from Property 4 (notice that t_0 can be chosen uniformly for k). The fourth and fifth terms of the right hand side can be arbitrary small uniformly for k for sufficiently large t_2 from Property 2. Lastly, the first term of the right hand side can be arbitrary small from the uniform convergence on the interval $[t_0, t_2]$. □

Note. The proof of Lemma 1 is for single-input single-output systems. An extension for multi-input multi-output systems is straightforward by applying the same argument elementwise.

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