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Youla-Kucera Parameterization for Nonlinear Systems via Observer Based Kernel Representations†

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This paper is concerned with Youla-Kucera parameterization for a class of nonlinear systems via kernel representations. Kernel representations are generalized left factorizations for nonlinear systems which are introduced by Paice et al. We give a rather natural generalization of Youla-Kucera parameterization by using observer based kernel representations. Furthermore we derive a state-space formula of the parameterization in this framework.

Key Words: nonlinear control, Youla-Kucera parameterization, kernel representation, coprime factorization

1. Introduction

Coprime factorization approach is widely used for analysis and synthesis of linear control systems. In the last decade, inspired by 1), a lot of research has been done on nonlinear extension of coprime factorizations [2)∼9)]. Input-output operators are used to express nonlinear systems and their factorizations, while transfer functions are used in the linear case. Nonlinear right coprime factorizations are natural extension of the linear ones and many satisfactory results were obtained [6),8),9)] which are very similar to the linear case results. Nonlinear left coprime factorizations can be used to derive the parameterization of stabilizing controllers and many results on this topic were obtained so far [1),2),7)]. However, nonlinear left coprime factorizations can not be well defined to have the consistency with right coprime factorizations.

Recently, kernel representations have been introduced [3),5)] as generalization of nonlinear left coprime factorizations which have the consistency with right coprime factorizations. It is also shown that the state-space realizations of kernel representations are computable, while it is very hard to obtain the state-space realizations of left factorizations. Furthermore the parameterization of all stabilizing controllers in input-output setting was obtained via kernel representations [5)]. However the parameterization in input-output setting intrinsically contains a difficulty in constructing its state-space realization. In input-output setting, two systems that have same realizations with different initial conditions are regarded as two different operators, because they have different input-output mappings. Therefore, when we construct a state-space realization of the parameterization, the controller has to know the initial state of the plant in order to realize the same input-output mapping of the plant itself. This problem is caused by the fact that any operators which have the same realizations cannot be identified in the context of input-output approach. The present paper proposes to associate such operators that have the same realizations by the notion of detectability which state observers should possess. The authors believe that this is the first result on the parameterization of all stabilizing controllers with state-space realizations by factorization approach. Furthermore, the relationship between input-output approach and the existing state-space approach [10)∼12)] is clarified as a consequence of the main result.

2. Notations

2.1 Signal space and operator stability

Any signal z is an element of its signal space $\mathcal{Z}$. The space $\mathcal{Z}$ is usually taken to be a set of functions from a time domain to a Euclidean vector space, e.g., $\mathcal{Z} = L_{pe}^m$.

An operator Σ with an input signal space $\mathcal{U}$, an output signal space $\mathcal{Y}$ and an initial condition $x^0 \in \mathcal{X}^0$ is denoted by $\Sigma^{x^0} : \mathcal{U} \to \mathcal{Y}$. Suppose the *well-definedness* and *stability* of operators are defined so that the set of well-defined or stable operators makes a ring with respect to the sum

$$(\Sigma + \Gamma)^{(x_\Sigma^0, x_\Gamma^0)}(u) := \Sigma^{x_\Sigma^0}(u) + \Gamma^{x_\Gamma^0}(u), \quad \forall u \in \mathcal{U}$$

with $\Sigma : \mathcal{U} \to \mathcal{Y}$ and $\Gamma : \mathcal{U} \to \mathcal{Y}$, and the product

$$(\Sigma \circ \Theta)^{(x_\Sigma^0, x_\Theta^0)}(z) := \Sigma^{x_\Sigma^0}(\Theta^{x_\Theta^0}(z)), \quad \forall z \in \mathcal{Z}$$

with $\Sigma : \mathcal{U} \to \mathcal{Y}$ and $\Theta : \mathcal{Z} \to \mathcal{U}$. That is, $(\Sigma + \Gamma)$ and $(\Sigma \circ \Theta)$ are well-defined (or stable) if Σ, Γ and Θ are well-defined (or stable); the identity operator Id and the zero operator 0 are stable. For example, causality, smoothness and Lipschitz continuity can be taken as the well-definedness of the operator, and typical input-output stability definitions such as L_p-stability, L_p finite gain stabil-

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ity and bounded-input bounded-output stability (BIBO) can be adopted as the stability.

An operator $\Sigma : \mathcal{U} \to \mathcal{Y}$ is said to be *invertible* if it has a well-defined inverse operator $\Sigma^{-1} : \mathcal{Y} \to \mathcal{U}$. An invertible operator $\Sigma : \mathcal{U} \to \mathcal{Y}$ is said to be *unimodular* if both Σ and Σ^{-1} are stable.

Furthermore a particular notation is adopted in this paper. The initial state x^0 of a system Σ^{x^0} is sometimes omitted and just denoted by Σ instead. When such a simplified notation is employed, $\bar{\Sigma}$ denotes a copy of a system Σ with a different initial condition, i.e., $\bar{\Sigma} = \Sigma^{x^1}$ while $\Sigma = \Sigma^{x^0}$.

2.2 State-space realizations

In order to obtain state-space results, we implicitly assume that any operator Σ^{x^0} has a state space realization

$$\Sigma^{x^0} : \left\{ \begin{array}{rcl} \dot{x} & = & f(x,u) \quad x(0) = x^0 \\ y & = & h(x,u) \end{array} \right. \tag{1}$$

with smooth functions f and h satisfying $f(0,0) = 0$ and $h(0,0) = 0$. When the state-space results are discussed, the operator stability should include that of the state behavior such as input-to-state stability.

2.3 Kernel representations

This subsection introduces kernel representations [3)∼5)] as generalization of left factorizations of linear systems.

Definition. A *kernel representation* of an operator $\Sigma^{x^0} : \mathcal{U} \to \mathcal{Y}$ is an operator $R_\Sigma^{x^0} : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}$ such that

$$y = \Sigma^{x^0}(u) \quad \Leftrightarrow \quad R_\Sigma^{x^0}(u,y) = 0 \tag{2}$$

holds for $\forall x^0 \in \mathcal{X}^0$, $\forall u \in \mathcal{U}$ and $\forall y \in \mathcal{Y}$. Further, a kernel representation $R_\Sigma^{x^0} : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}$ is said to be *well-defined* if there exists the pseudo-inverse operator $(R_\Sigma^{x^0})^\# : \mathcal{U} \times \mathcal{Z} \to \mathcal{Y}$ such that

$$y = (R_\Sigma^{x^0})^\#(u,z) \quad \Leftrightarrow \quad R_\Sigma^{x^0}(u,y) = z$$

holds for $\forall x^0 \in \mathcal{X}^0$, $\forall u \in \mathcal{U}$, $\forall y \in \mathcal{Y}$ and $\forall z \in \mathcal{Z}$. A notation $\Sigma_z^{x^0}$ is also used as shorthand for $(R_\Sigma^{x^0})^\#(\cdot, z)$.

Kernel representations are natural generalizations of left factorizations, because if an operator $\Sigma^{x^0} : \mathcal{U} \to \mathcal{Y}$ has a left factorization $\Sigma = \tilde{M}^{-1} \circ \tilde{N}$ with $\tilde{N}^{x^0} : \mathcal{U} \to \mathcal{Z}$ and $\tilde{M}^{x^0} : \mathcal{Y} \to \mathcal{Z}$ then a well-defined kernel representation of Σ is given by

$$R_\Sigma^{x^0}(u,y) = \ -\tilde{N}^{x^0}(u) + \tilde{M}^{x^0}(y) \ . \tag{3}$$

Namely the well-definedness corresponds to the invertibility of $\tilde{M}$ in the left factorization. The kernel representation of the linear case $\Sigma = \Sigma_\ell(s) = \tilde{M}(s)^{-1}\tilde{N}(s)$ is depicted in **Fig. 1**. Similar to Fig. 1, we depict a kernel representation $R_\Sigma^{x_\Sigma^0}$ as in **Fig. 2**.

Kernel representations are not equivalent to left factorizations because in general a kernel representation R_Σ is

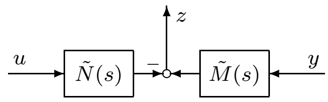

Fig. 1 The kernel representation of Σ_ℓ

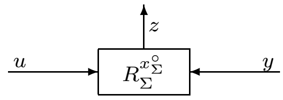

Fig. 2 The kernel representation of Σ

not always "separable", namely, it cannot be divided into two operators $\tilde{N}$ and $\tilde{M}$ as in (3).

Definition. A stable kernel representation $R_\Sigma^{x^0} : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}$ is said to be *coprime* if there exists a stable operator $X^{x^0} : \mathcal{Z} \to \mathcal{U} \times \mathcal{Y}$ such that

$$R_\Sigma^{x^0} \circ X^{x^0} = \mathrm{Id} \tag{4}$$

holds for $\forall x^0 \in \mathcal{X}^0$.

Equation (4) reduces to

$$-\tilde{N} \circ X_1 + \tilde{M} \circ X_2 = \mathrm{Id}$$

when R_Σ specializes to (3). Therefore the equation (4) is a natural generalization of the Bezout identity in the linear case.

2.4 Kernel representations of feedback systems

Consider a feedback system as shown in **Fig. 3**. Such a feedback system that interconnects $G^{x_G^0} : \mathcal{U} \to \mathcal{Y}$ and $K^{x_K^0} : \mathcal{Y} \to \mathcal{U}$ is denoted by $\{G^{x_G^0}, K^{x_K^0}\}$ or just $\{G, K\}$. Well-posedness and stability of the feedback system $\{G, K\}$ are investigated.

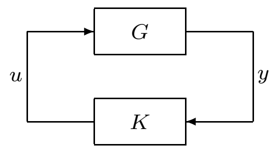

Fig. 3 Closed-loop system $\{G, K\}$

Suppose $G : \mathcal{U} \to \mathcal{Y}$ and $K : \mathcal{Y} \to \mathcal{U}$ have kernel representations $R_G : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_G$ and $R_K : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_K$

$$R_G^{x_G^0} \ : \ (u,y) \mapsto z_G \tag{5}$$

$$R_K^{x_K^0} \ : \ (y,u) \mapsto z_K. \tag{6}$$

Then the kernel representation $R_{\{G,K\}} : \mathcal{W} \to \mathcal{Z}_{GK}$ of the feedback system $\{G, K\}$ can be defined by

$$R_{\{G,K\}}^{(x_G^0, x_K^0)}(w) := \begin{pmatrix} R_K^{x_K^0}(y,u) \\ R_G^{x_G^0}(u,y) \end{pmatrix} = z_{GK}.$$

Here w and z_{GK} are condensed notations of (u,y) and (z_K, z_G) respectively, i.e.,

$$
\begin{array}{lcllc}
\mathcal{W} & \ni & w := (u, y) & \in & \mathcal{U} \times \mathcal{Y} \\
\mathcal{Z}_{GK} & \ni & z_{GK} := (z_K, z_G) & \in & \mathcal{Z}_K \times \mathcal{Z}_G \\
\mathcal{X}^0_{GK} & \ni & x^0_{GK} := (x^0_G, x^0_K) & \in & \mathcal{X}^0_G \times \mathcal{X}^0_K.
\end{array} \quad (7)
$$

Regarding the feedback system $\{G, K\}$ as a null-input system $w = \{G, K\}$ with w the output, we obtain

$$w = \{G, K\} \quad \Leftrightarrow \quad R_{\{G,K\}}(w) = 0 \quad (8)$$

which is the definition of the kernel representation (2). The null well-posedness and null internal stability of the feedback system $\{G, K\}$ are defined using the kernel representation $R_{\{G,K\}}$.

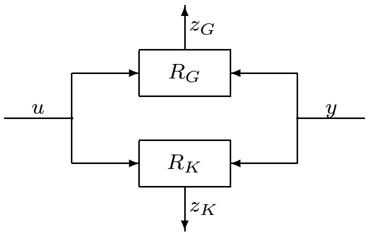

Fig. 4 The kernel representation of $\{G, K\}$

Next we consider the feedback system with the kernel representations R_G and R_K as depicted in **Fig. 4**. The null well-posedness and null internal stability are defined as follows.

Definition. A feedback system $\{G, K\}$ with a kernel representation $R^{(x^0_G, x^0_K)}_{\{G,K\}}$ is said to be *null well-posed* if the operator $R^{(x^0_G, x^0_K)}_{\{G,K\}}$ is invertible. Further, a null well-posed feedback system $\{G, K\}$ with a stable kernel representation $R^{(x^0_G, x^0_K)}_{\{G,K\}}$ is said to be *null internally stable* if $R^{(x^0_G, x^0_K)^{-1}}_{\{G,K\}}$ is stable.

Remark 1. The internal stability of a system $\{G, K\}$ with a kernel representation $R_{\{G,K\}}$ is equivalent to the coprimeness of $R_{\{G,K\}}$, because the unimodularity of $R_{\{G,K\}}$ is equivalent to the existence of a stable operator $X = R^{-1}_{\{G,K\}}$ such that

$$R_{\{G,K\}} \circ X = \mathrm{Id}.$$

This equation is a generalization of the double Bezout identity.

3. Parametrization of stabilizing controllers

3.1 Preliminaries

Paice and van der Schaft already gave the following result as a parameterization of all stabilizing plant-controller pairs via kernel representations [4),5)]. This result gives a parameterization of all null internally stabilizing pairs $\{G_S, K_Q\}$ in the input-output setting without the effect of the initial states.

Theorem 1. [5)] Suppose BIBO is adopted as the operator stability. Consider a null internally stable system $\{G, K\}$ with a kernel representation $R_{\{G,K\}}$, and a system Q with a stable kernel representation $R_Q : \mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{Z}_Q$, giving K_Q with the following stable kernel representation.

$$R_{K_Q} := R_Q \circ R_{\{G,K\}} \quad (9)$$

Then the feedback system $\{G, K_Q\}$ with the kernel representation $R_{\{G,K_Q\}}$ is null internally stable if and only if Q is stable.

Furthermore, given a null internally stable system $\{G, K^\star\}$ with a kernel representation $R_{\{G,K^\star\}}$ where $R_{K^\star} : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_{K^\star}$, then there exists a stable system $Q^\star$ with a kernel representation $R_{Q^\star}$, such that $R_{Q^\star} : \mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{Z}_{K^\star}$, and $K_{Q^\star} = K^\star$ hold.

The parameterization (9) can be depicted as in **Fig. 5** where R_Q is the free parameter.

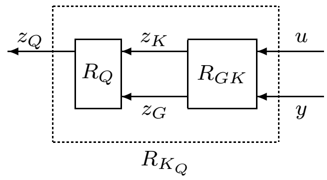

Fig. 5 The kernel representation of K_Q

In Theorem 1, the kernel representation R_{K_Q} defined by (9) contains the operator R_G. Hence the state-space realization of K_Q has to contain the state of the plant G, and this parameterization is not useful in a practical situation. Strictly speaking, this result does not give the stabilizing plant-controller pairs in the sense of Definition in section 2.4, because $R_{\{G_S,K_Q\}}$ is not necessarily unimodular for $\forall x^0_{G_S} \in \mathcal{X}^0_{G_S}$ and $\forall x^0_{K_Q} \in \mathcal{X}^0_{K_Q}$ unless the initial conditions $x^0_{G_S} = (x^0_S, x^0_G, x^0_K)$ and $x^0_{K_Q} = (x^0_Q, x^0_{\bar{G}}, x^0_{\bar{K}})$ satisfy $x^0_{\bar{G}} \equiv x^0_G$ and $x^0_{\bar{K}} \equiv x^0_K$ (which is equivalent to the equations $x_{\bar{G}}(t) \equiv x_G(t)$ and $x_{\bar{K}}(t) \equiv x_K(t)$ for $\forall t$ in their state-space realizations).

Most of the existing results on the parameterization of stabilizing controllers using the input-output approach have a similar difficulty, e.g., 2), 7). The main reason of this difficulty comes from the fact that the parameterization does not use the property of state observers whereas the linear parameterization is based on it. The characterization of kernel representations (or left coprime factorizations) which have the property of state observers can be found only in the reference 13) as long as the authors know, but it is not a satisfactory property for the parameterization. The following section proposes to use *detectable* kernel representations, which are based on state observers, in order to avoid the difficulty and derives a parameterization of all stabilizing controllers in the usual sense.

3. 2 Parametrization via detectable kernel representations

For the purpose stated in the previous subsection, observer based kernel representations are introduced. As its preparation, let us remember the linear case.

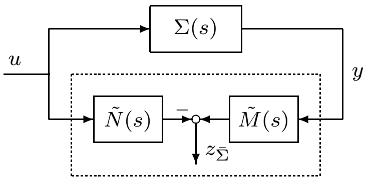

Fig. 6 Relationship between an observer and a left coprime factorization for a linear system $\Sigma(s)$

If an operator Σ is a linear system, then a stable kernel representation R_Σ of Σ can be constructed using its left coprime factorization. A left coprime factorization of Σ is closely related to its state observer. Actually, the state-space realization of R_Σ is a state observer of Σ as shown in **Fig. 6**. Moreover it is also an estimator of the external signal z_Σ added to $R_\Sigma^{\#}(\cdot, z_\Sigma)$ as in **Fig. 7**.

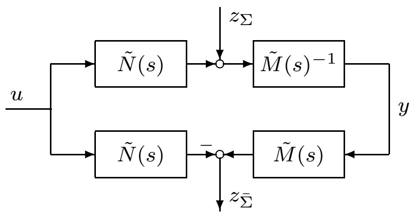

Fig. 7 The left coprime factor representation of a linear system $\Sigma(s)$ and its observer

This is the reason why the parameterization in Theorem 1 works well in the state-space setting in the linear case.

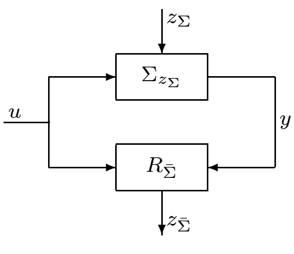

Fig. 8 Observer based kernel representation

Now we define detectable kernel representations which have the property discussed above. **Fig. 8** shows the concept of the detectability of R_Σ. Here Σ is a mapping: $\mathcal{U} \to \mathcal{Y}$. $R_{\bar{\Sigma}}$ is a copy of R_Σ with a different initial condition as stated in section 2. 1. We claim that $R_{\bar{\Sigma}}$ should have the property of an estimator of the external signal z_Σ added to $R_\Sigma^{\#}(\cdot, z_\Sigma)$ in the figure, and that if we employ kernel representations which have such a property then a parameterization similar to Theorem 1 holds in the usual sense of Definition in section 2. 4. The precise definition is stated as follows:

Definition. A kernel representation $R_\Sigma^{x_\Sigma^0} : \mathcal{W} \to \mathcal{Z}_\Sigma$ is said to be *detectable* if the operator without input

$$\partial R_{\Sigma(w)}^{(x_\Sigma^1, x_\Sigma^2)}(0) \tag{10}$$

is stable irrespective of $w \in \mathcal{W}$, where ∂ is the differential operator defined by

$$\partial \Gamma_{(w)}^{(x^1, x^2)}(v) := \Gamma^{x^1}(w+v) - \Gamma^{x^2}(w). \tag{11}$$

A kernel representation R_Σ is said to be detectable if the difference of the outputs z_Σ's from two R_Σ's are close to each other for any signal $w \in \mathcal{W}$, so this is not a trivial property from the stability of R_Σ. Detectability is a natural property of the z_Σ-estimator which appears in Fig. 8 because two same R_Σ's should estimate the signals z_Σ's close to each other.

For example, if L_p-stability with the signal space $\mathcal{Z} = L_{pe}$ is taken as the operator stability, then the detectability definition (10) of R_Σ reduces to

$$R_\Sigma^{x_\Sigma^1}(w) - R_\Sigma^{x_\Sigma^2}(w) \in L_p, \quad \forall w \in L_{pe}, \forall x_\Sigma^1, \forall x_\Sigma^2 \in \mathcal{X}_\Sigma^0.$$

Or, if L_p finite gain stability is taken as the operator stability, then the detectability of R_Σ reduces to

$$\|R_\Sigma^{x_\Sigma^1}(w) - R_\Sigma^{x_\Sigma^2}(w)\|_p \le \phi(x_\Sigma^1, x_\Sigma^2),$$
$$\forall w \in L_{pe}, \forall x_\Sigma^1, \forall x_\Sigma^2 \in \mathcal{X}_\Sigma^0$$

with a smooth function ϕ satisfying $\phi(0,0) = 0$.

Remark 2. The definition of detectability introduced in the original version of the paper was defined only for BIBO stability where the property

$$R_\Sigma^{x_\Sigma^1}(w) \in \mathcal{Z}_\Sigma^s \quad \Leftrightarrow \quad R_\Sigma^{x_\Sigma^2}(w) \in \mathcal{Z}_\Sigma^s, \quad \forall w \in \mathcal{W}$$

with the bounded signal space $\mathcal{Z}_\Sigma^s \subset \mathcal{Z}_\Sigma$ is taken as the definition of detectability. This characterization of detectability is slightly less restrictive than the definition taken here if BIBO is taken as the operator stability.

Assuming the detectability of R_G the kernel representation of the plant, the following result can be obtained.

Theorem 2. Consider a null internally stable system $\{G, K\}$ with a kernel representation $R_{\{G,K\}}$ such that R_G is detectable, and system Q with a well-defined stable kernel representation $R_Q : \mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{Z}_Q$, giving K_Q with the stable kernel representation

$$R_{K_Q}^{(x_Q^0, x_G^1, x_K^0)} := R_Q^{x_Q^0} \circ R_{\{G,K\}}^{(x_G^1, x_K^0)}$$

$$\text{s.t. } R_{K_Q} \text{ is well-defined} \tag{12}$$

where the true initial condition of R_G is x_G^0. Then the feedback system $\{G, K_Q\}$ with the kernel representation $R_{\{G,K_Q\}}$ is null internally stable if and only if it is null well-posed and $R_Q^{\#}$ is stable.
Furthermore, given a null internally stable system $\{G, K^\star\}$ with a kernel representation $R_{\{G,K^\star\}}$ where $R_{K^\star} : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_{K^\star}$, then there exists a well-defined stable kernel representation $R_{Q^\star} : \mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{Z}_{K^\star}$, such that $K_{Q^\star} = K^\star$ holds and $R_{Q^\star}^{\#}$ is stable.

Proof. The proof can be obtained by setting $e_{12} = 0$ in the proof of Theorem 3 shown later. □

When G, K and Q are linear operators and they have left factorizations $G(s) = \tilde{M}(s)^{-1}\tilde{N}(s)$ and $K(s) = \tilde{V}(s)^{-1}\tilde{U}(s)$ respectively, then solve $R_{K_Q} = 0$ with the equation (12) in Theorem 2 derives the following explicit parameterization of K_Q

$$K_Q(s) = (\tilde{V}(s)+Q(s)\tilde{N}(s))^{-1}(\tilde{U}(s)+Q(s)\tilde{M}(s)) \tag{13}$$

with Q s.t. $(\tilde{V}(s) + Q(s)\tilde{N}(s))$ is invertible, which coincides with the linear Youla-Kucera parameterization.

3.3 State-space realization

This subsection discusses the state-space realization of the parameterization given in Theorem 2. Consider a plant G with a state-space realization

$$G : \begin{cases} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{cases} \tag{14}$$

Suppose there exists a *detectable* kernel representation R_G and a controller K with R_K which null internally stabilizes G with R_G:

$$R_G : \begin{cases} \dot{x}_G &= f_G(x_G,u,y) \\ z_G &= h_G(x_G,u,y) \end{cases} \tag{15}$$

$$R_K : \begin{cases} \dot{x}_K &= f_K(x_K,y,u) \\ z_K &= h_K(x_K,y,u) \end{cases} \tag{16}$$

Then all (null internally) stabilizing controllers are given by Theorem 2 as follows.

Corollary 1. Consider G with the state-space realization (14) and its kernel representation R_G in (15). Suppose R_G is detectable and that there exists a controller K with R_K in (16) such that $\{G, K\}$ with $R_{\{G,K\}}$ is null internally stable. Then every null internally stabilizing controller K_Q with a kernel representation R_{K_Q} is parametrized by

$$K_Q : \begin{cases} \dot{x}_K = f_K(x_K, y, h_{K_Q}(x_{\bar{G}K} x_Q, y)) \\ \dot{x}_{\bar{G}} = f_G(x_{\bar{G}}, h_{K_Q}(x_{\bar{G}K}, x_Q, y), y) \\ \dot{x}_Q = f_Q(x_Q, h_G(x_{\bar{G}}, h_{K_Q}(x_{\bar{G}K}, x_Q, y), y)) \\ \;u = h_{K_Q}(x_{\bar{G}K}, x_Q, y) \end{cases} \tag{17}$$

where the parameter $Q : \mathcal{Z}_{\bar{G}} \to \mathcal{Z}_K$

$$Q : \begin{cases} \dot{x}_Q &= f_Q(x_Q, z_{\bar{G}}) \\ z_K &= h_Q(x_Q, z_{\bar{G}}) \end{cases} \tag{18}$$

is any stable operator and the set of output functions $\{h_G, h_K, h_Q\}$ is supposed to have a unique solution with respect to u as follows.

$$u = h_{K_Q}(x_{\bar{G}K}, x_Q, y) \tag{19}$$

The parameterization in Corollary 1 can be depicted as in **Fig. 9**, which can be explained as follows: The operator $R_{\bar{G}}$ is a state observer (or a disturbance estimator) of G_{z_G} where its output $z_{\bar{G}}$ describes the external disturbance z_G when $z_G \neq 0$, or the state observing error when $z_G = 0$. Q is the stable free parameter and K_{z_K} is a stabilizing controller with an external (reference) input z_K. This figure reveals that this result is a natural extension of the linear Youla-Kucera parameterization.

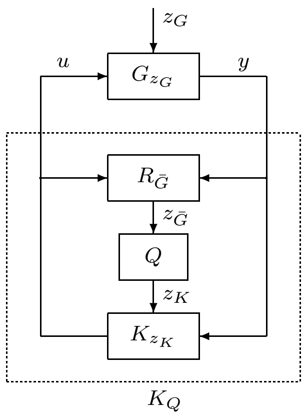

Fig. 9 The construction of the parameterization

3.4 Further investigation on state-space realization

Now we show an example of the state-space realization of the feedback system $\{G, K\}$ with kernel representation $R_{\{G,K\}}$ satisfying the assumptions in Corollary 1. We adopt the stability as BIBS (Bounded-Input Bounded-State) stability and $\mathcal{Z}^s$ denotes the bounded (stable) subset of the signals in $\mathcal{Z}$. Consider an operator G with a state-space realization:

$$G : \begin{cases} \dot{x} &= f(x,u) \\ y &= h(x) \end{cases} \tag{20}$$

Let us employ the following assumptions:

(A1) There exists a state observer of G as

$$\dot{x} = f_G(x,u,y) \tag{21}$$

where $(u,y) \in \mathcal{U}^s \times \mathcal{Y}^s \;\Rightarrow\; x \in \mathcal{X}^s$. Here we consider another observer with the same realization

$$\dot{\bar{x}} = f_G(\bar{x},u,y) \tag{22}$$

and apply the same input $(u, y) \in \mathcal{U} \times \mathcal{Y}$ to them, then $(x - \bar{x}) \in \mathcal{X}^s$ holds.

(A2) There exists a function h_G satisfying

$$y = h_G(x, z_G) \tag{23}$$

and there also exists its pseudo inverse $z_G = h_G^{\#}(x, y)$ satisfying $h_G(x, 0) \equiv h(x)$. Furthermore, $(x - \bar{x}) \in \mathcal{X}^s \Rightarrow h_G^{\#}(\bar{x}, h(x)) \in \mathcal{Z}_G^s$ holds.

(A3) There exists a stabilizing control for (21) such that

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} h_K(x + x_e, y, z_K) \\ h_G(x, z_G) \end{pmatrix} \tag{24}$$

Here the feedback system with the plant (21) and the above controller satisfies

$$(z_K, z_G) \in \mathcal{Z}_K^s \times \mathcal{Z}_G^s \quad \Leftrightarrow \quad (u, y) \in \mathcal{U}^s \times \mathcal{Y}^s \tag{25}$$

for all $x_e \in \mathcal{X}^s$ and there exists a function $h_K^{\#}$ satisfying $z_K = h_K^{\#}(x + x_e, y, u)$.

Proposition. Consider the operator G in (20). Suppose (A1), (A2) and (A3) hold. Then a pair R_G and R_K satisfying the assumptions in Theorem 2 (and Corollary 1) is given by

$$R_G : \begin{cases} \dot{x}_G &= f_G(x_G, u, y) \\ z_G &= h_G^{\#}(x_G, y) \end{cases} \tag{26}$$

$$R_K : \begin{cases} \dot{x}_K &= f_G(x_K, u, y) \\ z_K &= h_K^{\#}(x_K, y, u) \end{cases} \tag{27}$$

Proof. The proposition is straightforwardly obtained by checking the conditions in Theorem 2. □

4. Parametrization in the presence of additive disturbances

4.1 Internal stability

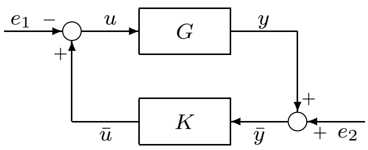

Fig. 10 The feedback system $\{G, K\}$ with additive disturbances

Consider the feedback system depicted in **Fig. 10** here. We use the condensed notations as in (7) if no confusion arises. The stability of the feedback system $\{G, K\}$ with additive disturbances as in Fig. 10 is considered. Such a configuration is often treated in the literature on right coprime factorizations, e.g., 8). Let us define a new operator $E_{\{G,K\}}^{(x_G^0, x_K^0)} : \mathcal{E}_{12} \to \mathcal{W}$ which is a mapping from the external additive signal (e_1, e_2) to the loop signal (u, y) in Fig. 10.

$$E_{\{G,K\}}^{(x_G^0, x_K^0)}(e_{12}) := \left(\begin{pmatrix} -\mathrm{Id} & K^{x_K^0} \\ -G^{x_G^0} & \mathrm{Id} \end{pmatrix}^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \right) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Definition. A feedback system $\{G, K\}$ is said to be *well-posed* if the operator $E_{\{G,K\}}^{(x_G^0, x_K^0)}$ exists and is well-defined. Further, a well-posed feedback system $\{G, K\}$ is said to be *internally stable* if $E_{\{G,K\}}^{(x_G^0, x_K^0)}$ is stable.

4.2 Strong internal stability

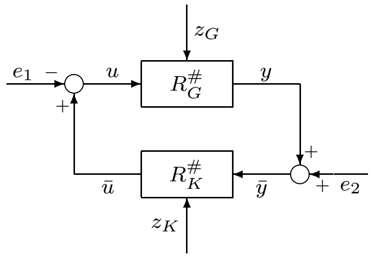

Fig. 11 Strong well-posedness and strong internal stability of $\{G, K\}$

Employing the kernel representations R_G and R_K as in (5) and (6), a kernel representation of the operator $E_{\{G,K\}}$ can be defined as

$$R_{E_{\{G,K\}}}^{(x_G^0, x_K^0)}(e_{12}, w) := \begin{pmatrix} R_K^{x_K^0}(w + e_{12}) \\ R_G^{x_G^0}(w) \end{pmatrix} = z_{GK}.$$

It is easy to see that

$$w = E_{\{G,K\}}(e_{12}) \quad \Leftrightarrow \quad R_{E_{\{G,K\}}}(e_{12}, w) = 0$$

which is the definition of the kernel representation (2). The relation (8) is the null input version of this equation. Strong well-posedness and strong internal stability are defined as follows:

Definition. A feedback system $\{G, K\}$ with a kernel representation $R_{\{G,K\}}^{(x_G^0, x_K^0)}$ is said to be *strongly well-posed* if $R_{E_{\{G,K\}}}^{(x_G^0, x_K^0)}$ is well-defined. Further, a strongly well-posed feedback system $\{G, K\}$ with a stable kernel representation $R_{\{G,K\}}^{(x_G^0, x_K^0)}$ is said to be *strongly internally stable* if $R_{E_{\{G,K\}}}^{(x_G^0, x_K^0)\#}$ is stable.

The concepts of strong well-posedness and strong internal stability are shown in **Fig. 11**. We abuse the notations $R_G^{\#}$ and $R_K^{\#}$ to denote operators that might not be well-defined.

4.3 Youla-Kucera parameterization

The detectability of a kernel representation is extended to fit the strong internal stability.

Definition. A kernel representation $R_\Sigma^{x_\Sigma^0} : \mathcal{W} \to \mathcal{Z}_\Sigma$ is said to be *strongly detectable* if the operator

$$\partial R_{\Sigma(w)}^{(x_\Sigma^1, x_\Sigma^2)} \tag{28}$$

is stable irrespective of $w \in \mathcal{W}$, where ∂ is the differential operator defined in (11).

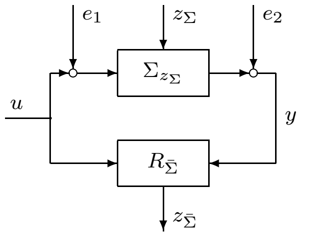

Fig. 12 Strong detectability

This definition implies that R_Σ is a z_Σ-estimator of the system $R_\Sigma^{\#}(\cdot, z_\Sigma)$ which works well even when the input w has an additive disturbance as in **Fig. 12**. For example, if L_p-stability with the signal space $\mathcal{Z} = L_{pe}$ is taken as the operator stability, then the strong detectability definition (28) of R_Σ reduces to

$$w - v \in L_p \quad \Rightarrow \quad R_\Sigma^{x_\Sigma^1}(w) - R_\Sigma^{x_\Sigma^2}(v) \in L_p$$
$$\forall w, \forall v \in L_{pe}, \forall x_\Sigma^1, \forall x_\Sigma^2 \in \mathcal{X}_\Sigma^0.$$

Or, if L_p finite gain stability is taken as the operator stability, then the strong detectability of R_Σ reduces to

$$\|R_\Sigma^{x_\Sigma^1}(w) - R_\Sigma^{x_\Sigma^2}(v)\|_p \leq \gamma \|w - v\|_p + \phi(x_\Sigma^1, x_\Sigma^2)$$
$$\forall w, \forall v \in L_{pe}, \forall x_\Sigma^1, \forall x_\Sigma^2 \in \mathcal{X}_\Sigma^0$$

with a scalar γ and a smooth function ϕ satisfying $\phi(0,0) = 0$.

The concept of strong detectability gives us the following result.

Theorem 3. Consider a strongly internally stable system $\{G, K\}$ with a kernel representation $R_{\{G,K\}}$ such that R_G is strongly detectable, and system Q with a well-defined stable kernel representation $R_Q : \mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{Z}_Q$, giving K_Q with the stable kernel representation

$$R_{K_Q}^{(x_Q^0, x_G^1, x_K^0)} := R_Q^{x_Q^0} \circ R_{\{G,K\}}^{(x_G^1, x_K^0)}$$
$$\text{s.t. } R_{K_Q} \text{ is well-defined} \tag{29}$$

where the true initial condition of R_G is x_G^0. Then the feedback system $\{G, K_Q\}$ with the kernel representation $R_{\{G,K_Q\}}$ is strongly internally stable if and only if it is strongly well-posed and $R_Q^{\#}$ is stable.

Furthermore, given a strongly internally stable system $\{G, K^\star\}$ with a kernel representation $R_{\{G,K^\star\}}$ where $R_{K^\star} : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_{K^\star}$, then there exists a well-defined stable kernel representation $R_{Q^\star} : \mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{Z}_{K^\star}$, such that $K_{Q^\star} = K^\star$ holds and $R_{Q^\star}^{\#}$ is stable.

Proof. As mentioned in section 2.1, we use the notation $z_{\bar{G}} = R_{\bar{G}}(u, y)$ in order to describe a copy of $z_G = R_G(u, y)$ in the parametrized controller. The former part, i.e., the sufficiency is proved. Firstly, from the internal stability of the system $\{G, K\}$, we have a stable mapping $z_{\bar{G}K} \mapsto w$

$$w = R_{\{\bar{G},K\}}^{-1}(z_{\bar{G}K}). \tag{30}$$

Secondly, from the stability of $R_Q^{\#}$, we have another stable mapping $z_{\bar{G}Q} \to z_{\bar{G}K}$

$$z_{\bar{G}K} = R_{SQ}^{-1}(z_{\bar{G}Q}) = \begin{pmatrix} R_Q^{\#}(z_{\bar{G}}, z_Q) \\ z_{\bar{G}} \end{pmatrix} \tag{31}$$

with the trivial kernel representation

$$z_S = R_S(z_K, z_G) := z_G.$$

Furthermore we have a mapping $z_G \mapsto z_{\bar{G}}$ parametrized by w

$$\begin{aligned} z_{\bar{G}} &= z_G + (z_{\bar{G}} - z_G) = z_G + R_{\bar{G}}(\bar{w}) - R_G(w) \\ &= z_G + \partial R_{G(w)}(e_{12}) \end{aligned} \tag{32}$$

which is stable irrespective of $w \in \mathcal{W}$. Therefore, it follows from the assumption of the strong well-posedness of the system $\{G, K_Q\}$ and the equations (30), (31) and (32) that $R_{E\{G,K_Q\}}^{\#}$ can be described by the following stable mapping:

$$\begin{aligned} w &= R_{E\{G,K_Q\}}^{\#}(e_{12}, z_{GQ}) \\ &= R_{\{\bar{G},K\}}^{-1} \circ \begin{pmatrix} R_Q^{\#}(z_G + \partial R_{G(w)}(e_{12}), z_Q) \\ z_G + \partial R_{G(w)}(e_{12}) \end{pmatrix} \end{aligned} \tag{33}$$

Furthermore $R_Q^{\#}$ can be written as follows:

$$R_Q^{\#} = R_K \circ R_{\{\bar{G},K_Q\}}^{-1}.$$

Hence this proves the necessity and completes the former part.

Next, the latter part is discussed. Let $R_{Q^\star}$ defined by

$$R_{Q^\star} := R_{K^\star} \circ R_{\{\bar{G},K\}}^{-1}$$

Its pseudo-inverse mapping from $(z_{\bar{G}}, z_{K^\star})$ to z_K can be explicitly written by

$$z_K = R_K \circ R_{\{\bar{G},K^\star\}}^{-1}(z_{K^\star}, z_{\bar{G}}) =: R_{Q^\star}^{\#}(z_{\bar{G}}, z_{K^\star})$$

Hence $R_{Q^\star}$ is well-defined from the null well-posedness of the system $\{G, K^\star\}$. Then the rest of the theorem follows immediately. □

4.4 State-space realization

Here an investigation on the state-space realizations similar to section 3.4 is given. The same assumptions as in section 3.4 are made. Consider G in (20) and suppose there exists its state observer

$$\dot{x} = f_G(x, u', y) \tag{34}$$

satisfying the following assumption.

(B1) $(u, y) \in \mathcal{U}^s \times \mathcal{Y}^s \;\Rightarrow\; x \in \mathcal{X}^s$ holds and, for two copies of the same state observer

$$\dot{x} = f_G(x, u, y) \tag{35}$$
$$\dot{\bar{x}} = f_G(\bar{x}, u + e_1, y + e_2) \tag{36}$$

satisfy $(e_1, e_2) \in \mathcal{E}_1^s \times \mathcal{E}_2^s \;\Rightarrow\; (x - \bar{x}) \in \mathcal{X}^s$ for all $(u, y) \in \mathcal{U} \times \mathcal{Y}$.

Based on this state observer, we construct a (strongly internally) stabilizing controller K as

$$K : \left\{ \begin{array}{rcl} \dot{x}_K &=& f_G(x_K, k(x_K), y) \\ u' &=& k(x_K) \end{array} \right. \tag{37}$$

Here $(u', y) = (k(\bar{x}), h(\bar{x}))$ is a stabilizing controller for the state observer (34) in the sense that the feedback with inputs e_1 and e_2

$$\begin{pmatrix} u' \\ y \end{pmatrix} = \begin{pmatrix} k(\bar{x}) + e_1 \\ h(\bar{x}) + e_2 \end{pmatrix} \tag{38}$$

with the system (34) satisfies the following additional assumptions:

(B2) $(e_1, e_2) \in \mathcal{E}_1^s \times \mathcal{E}_2^s \;\Rightarrow\; (u', y) \in \mathcal{U}^s \times \mathcal{Y}^s$.

(B3) $x \in \mathcal{X}^s \;\Rightarrow\; (k(x), h(x)) \in \mathcal{U}^s \times \mathcal{Y}^s$.

(B4) $(x - \bar{x}) \in \mathcal{X}^s \;\Rightarrow\; (k(x) - k(\bar{x}), h(x) - h(\bar{x})) \in \mathcal{U}^s \times \mathcal{Y}^s$.

Under those assumptions, we can obtain the following result.

Proposition. Consider the operator G in (20). Suppose the assumptions (B1)–(B4) hold. Then a pair R_G and R_K satisfying the assumptions in Theorem 3 is given by

$$R_G : \left\{ \begin{array}{rcl} \dot{x}_G &=& f_G(x_G, u, y) \\ z_G &=& y - h(x_G) \end{array} \right. \tag{39}$$

$$R_K : \left\{ \begin{array}{rcl} \dot{x}_K &=& f_G(x_K, u, y) \\ z_K &=& u - k(x_K) \end{array} \right. \tag{40}$$

Proof. The assumptions (B1) and (B3) imply that R_G and R_K are stable well-defined kernel representations of G and K respectively, while (B1) and (B4) imply that R_G is strongly detectable. The state-space realization of the feedback system as in Fig. 11 is described by

$$\{G_{z_G}, K_{z_K}\} : \left\{ \begin{array}{l} \dot{x}_G \;=\; f_G(x_G, u, y - e_2) \\ \dot{x}_K \;=\; f_G(x_K, u - e_1, y) \\ \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} k(x_K) + z_K + e_1 \\ h(x_G) + z_G + e_2 \end{pmatrix} \end{array} \right. . \tag{41}$$

Therefore, (B1), (B2) and (B4) suggest

$$\begin{aligned} &(z_K, z_G, e_1, e_2) \in \mathcal{Z}_K^s \times \mathcal{Z}_G^s \times \mathcal{E}_1^s \times \mathcal{E}_2^s \\ &\Rightarrow (x_G - x_K) \in \mathcal{X}^s \\ &\Rightarrow \begin{pmatrix} k(x_K) - k(x_G) \\ h(x_G) - h(x_K) \end{pmatrix} \in \mathcal{U}^s \times \mathcal{Y}^s \\ &\Rightarrow (u, y) \in \mathcal{U}^s \times \mathcal{Y}^s. \end{aligned} \tag{42}$$

This implies the strong internal stability of $\{G, K\}$ with $R_{\{G,K\}}$ which completes the proof. □

The parameterization given above has a similar formulation to the linear case result.

5. Relationship to the existing results

This section discusses the relationship between the results obtained above and the existing results on the parameterization in state-space setting[10)~12)]. Those existing results are based on a weaker stability definition than that employed in the former part of this paper. Hence we define a weaker stability concept using kernel representations here.

Definition. A feedback system $\{G, K\}$ with a kernel representation $R_{\{G,K\}}$ is said to be *weakly stable* if both the mapping $z_K \mapsto u$ and its inverse $u \mapsto z_K$ in **Fig. 13** exist and are stable.

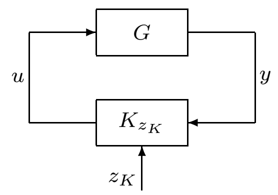

Fig. 13 Weak stability

Furthermore, the weak detectability is also defined in order to handle weak stability.

Definition. A kernel representation $R_\Sigma^{x_\Sigma^0} : \mathcal{W} \to \mathcal{Z}_\Sigma$ is said to be *weakly detectable* if the operator

$$\partial R^{(x_\Sigma^1, x_\Sigma^2)}_{\Sigma(u, \Sigma^{x_\Sigma^1}(u))}(0) \tag{43}$$

is stable where ∂ is the differential operator defined in (11).

Since the kernel representation $R_{\{G,K\}}$ of a weakly stable feedback system $\{G, K\}$ is not coprime (unimodular), we cannot obtain the results similar to those in the previous sections. Therefore, in general we can only parameterize a class of stabilizing controllers. However, if we employ a special stability definition, we can parameterize all locally stabilizing controllers in the same framework as shown in the following theorem. Input-to-state stability[6), 14), 15)] is employed here and its definition is given as follows.

Definition. The system Σ^{x^0} in (1) is said to be *input-to-state stable (ISS)* if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ satisfying

$$\|x(t)\| \;\le\; \beta(\,\|\,x^0\|,\; t\,) + \gamma(\,\|u\|_\infty) \tag{44}$$

Before stating the result, a remark on weak detectability is given.

Remark 3. Consider the system G in (20). Suppose the system G is weakly detectable in Vidyasagar's sense[16), 17)]. Then there exists an output-injection state observer of G described by

$$\dot{\bar{x}} = f_G(\bar{x}, u, y) \quad \bar{x}(0) = \bar{x}^0$$

and a $\mathcal{KL}$ function β such that the following inequality holds.

$$\|x(t) - \bar{x}(t)\| \le \beta(\|x^0 - \bar{x}^0\|, t), \quad \forall u \in \mathcal{U}$$

If G is weakly detectable, then R_G defined by

$$R_G : \begin{cases} \dot{x}_G &= f_G(x_G, u, y) \\ z_G &= h_G(x_G, y) \end{cases}$$

is weakly detectable, where $h_G : y(t) \mapsto z_G(t)$ is invertible function ($y = h_G^{\#}(x_G, z_G)$) such that $h_G(x, h(x)) \equiv 0$.

The result is now stated as follows.

Theorem 4. (i) Consider a weakly stable system $\{G, K\}$ with a kernel representation $R_{\{G,K\}}$ such that R_G is weakly detectable, and system Q with a well-defined stable kernel representation $R_Q : \mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{Z}_Q$, giving K_Q with the kernel representation (29). Then the feedback system $\{G, K_Q\}$ with the kernel representation $R_{\{G,K_Q\}}$ is weakly stable if it is null well-posed and $R_Q^{\#}$ is stable.

(ii) Consider any operator has a state-space realization as in (1). The stability definition of operators is taken to be ISS. Consider a weakly stable system $\{G, K^\star\}$ with a kernel representation $R_{\{G,K^\star\}}$ such that R_G is made as in Remark 3 (weakly detectable) where $R_{K^\star} : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_{K^\star}$, then there exists a stable kernel representation $R_{Q^\star}$: $\mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{Z}_{K^\star}$, such that $K_{Q^\star} = K^\star$ holds and $R_{Q^\star}^{\#}$ is locally stable, that is, the parameterization in (i) gives all locally stabilizing controllers.

Proof. The part (i) can be proved in a similar way to the sufficiency of Theorem 2. The part (ii) is now proved. Suppose the state-space realization of R_Q is given by

$$R_Q : \begin{cases} \dot{x}_Q &= f_Q(x_Q, z_G, z_K), \quad x_Q(0) = x_Q^0 \\ z_Q &= h_Q(x_Q, z_G, z_K) \end{cases}$$

and consider the case $x_G^0 = x_{\bar{G}}^0 = 0$, $x_K^0 = 0$ and $z_Q \equiv 0$. Then the following inequality holds for some $\mathcal{KL}$ function β and $\mathcal{K}$ function γ from the ISS property of the whole system $R_{\{G,K_Q\}}$.

$$\begin{aligned} \|x_Q(t)\| &\le \|(x_G(t), x_K(t), x_{\bar{G}}(t), x_Q(t))\| \\ &\le \beta(\|(x_G^0, x_K^0, x_{\bar{G}}^0, x_Q^0)\|, t) + \gamma(\|z_Q\|_\infty) \\ &= \beta(\|x_Q^0\|, t) \end{aligned}$$

Setting $z_{\bar{G}} \equiv 0$, the system

$$\begin{aligned} \dot{x}_Q &= f_Q(x_Q, z_{\bar{G}}, h_Q^{\#}(x_Q, z_{\bar{G}}, z_Q)) \\ &\equiv f_Q(x_Q, 0, h_Q^{\#}(x_Q, 0, 0)) \end{aligned}$$

is asymptotically stable in the Lyapunov sense. Hence there exists a Lyapunov function V_Q which locally satisfies the following relations by the converse Lyapunov theorem [17), 18)].

$$\alpha_1(\|x_Q\|) \le V_Q(x_Q) \le \alpha_2(\|x_Q\|)$$
$$\dot{V}_Q(x_Q) \le -\alpha_3(\|x_Q\|) + c_4\|z_{\bar{G}}\| + c_5\|z_Q\|$$

Here $\alpha_i(\cdot)$'s are $\mathcal{K}$ functions and c_j's are positive constants. These inequalities imply $R_Q^{\#}$ is locally stable (with small input), which completes the proof. □

As in the following remark, in Theorem 4, (ii) shows that the parameterization in (i) gives all local stabilizing controllers and it is equivalent to the existing result by state-space approach [12)]. Therefore Theorem 4 shows the relationship between input-output approach and state-space approach. The stability notion of ISS plays an important role to connect the two different frameworks.

Remark 4. Consider a weakly detectable system G in the form of (20) which has a state-observer as in Remark 3. Suppose moreover that there exists an R_K in the following form such that $R_{\{G,K\}}$ is weakly stable,

$$R_K : \begin{cases} \dot{x}_K &= f_G(x_K, u, y) \\ z_K &= h_K(x_K, y, u) \end{cases}$$

where $h_K : u(t) \mapsto z_K(t)$ is invertible ($u = h_K^{\#}(x_K, y, z_K)$). And choose Q as

$$Q : \begin{cases} \dot{q}_1 &= f_q(q_1, q_2, \hat{y}) \\ \dot{q}_2 &= f_G(q_2, h_K^{\#}(q_2, \hat{y}, z_K), \hat{y}) \\ z_K &= h_q(q_1, q_2, \hat{y}) \\ \hat{y} &:= h_G^{\#}(q_2, z_{\bar{G}}) \end{cases}$$

Then the following set of conditions is a necessary and sufficient condition for local stability of Q in Theorem 4.

- $\dot{q}_1 = f_q(q_1, 0, 0)$ is asymptotically stable.
- $f_q(q_1, x, h(x)) = f_q(q_1, 0, 0)$.
- $h_q(q_1, x, h(x)) = h_q(q_1, 0, 0)$.
- $h_q(0, 0, 0) = 0$.

These conditions are equivalent to 12) which gives the parameterization of all local stabilizing controllers. Thus Theorem 4 implies the consistency with the existing result. In addition, this result shows that the class of stabilizing controllers given in Theorem 4 is sufficiently large at least in the local setting.

6. Conclusion

This paper is concerned with a parameterization of stabilizing controllers. By employing observer based kernel representations, we can obtain a parameterization of all stabilizing controllers. The authors believe that this is the first result on the parameterization of all stabilizing controller in state-space setting based on nonlinear coprime factorizations (kernel representations). More precisely we investigate the relation between kernel representations and state observers in the state-pace realization and derives the observer properties for parameterization. Furthermore, we extend this result to the feedback systems in the presence of additive external disturbances, and also clarify the relationship between our input-output approach and the existing state-space approach to the parameterization of stabilizing controllers.

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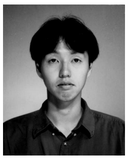
Kenji Fujimoto (Member)



Kenji Fujimoto was born in Osaka, Japan in 1971. He received his B.Sc. and M.Sc. degrees in engineering and his Ph.D. degree in informatics from Kyoto University, Japan, in 1994 and 1996 and 2001 respectively. From 1997 to 1998 he was a Research Associate of Graduate School of Engineering at Kyoto University, Japan. From 1999 to 2000 he was a Research Fellow of Department of Electrical Engineering at Delft University of Technology, The Netherlands. His current position is a Research Associate of Graduate School of Informatics at Kyoto University, Japan. His research interests include nonlinear systems theory and control.

Toshiharu Sugie (Member)

Toshiharu Sugie received his B.S., M.S. and Ph.D. degrees in engineering from Kyoto University, Kyoto, Japan, in 1976, 1978 and 1985, respectively. From 1978 to 1980, he was a Research Member of the Musashino Electric Communication Laboratory in NTT, Musashino, Japan. From 1984 to 1988, he was a Research Associate of Department of Mechanical Engineering at University of Osaka Prefecture, Osaka. In 1988, he joined Kyoto University, where he is currently a Professor of Department of Systems Science. His current research interests include robust control, nonlinear control, and system identification.

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