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A New Approach to LQ Design: Application to the Design of Optimal Servo Systems

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First, a new design theory of LQ regulator is developed from the practical viewpoint by applying some results on the inverse problem of LQ regulator. The LQ regulator considered here is, unlike the usual one, designed without specifying a performance cost, but it proves to be such a feedback control that minimizes some cost. This theory is then applied to design an optimal servosystem with the following features suitable for practical use: 1) a time constant of the output response can be taken as our design spec., 2) the design computation requires no Riccati solutions, 3) there is a close structural relationship between the controlled system and the resulting feedback matrix.

Key Words: : optimal control, inverse LQ problem, optimal servosystem

1. Introduction

Despite the recent increase of practical applications of LQ regulator, this design methodology still suffers from various difficulties in practical use such as 1) implicit relationship between the quadratic weights to be selected and the resulting closed-loop responses, 2) complicated and time-consuming design computation based on Riccati solutions, 3) complex configuration of the control system using almost full-state feedback of the plant structure. The LQ regulator is, however, known, irrespective of the choice of weights, for its desirable properties such as reduced sensitivity and robust stability [1]. In addition, of practical importance are the properties of the regulator to be designed, rather than the weights to be selected. From this viewpoint, it may be more practical in the LQ regulator design to give up the weight selection and design instead those state feedback controls that are optimal for some unknown weights, thereby simplifying the design procedures. This motivates us to design an LQ regulator from the viewpoint of the inverse regulator problem [2], which is a new approach to the LQ design that we provide in this paper. Here we develop such an LQ design theory by utilizing some pertinent results on the inverse regulator problem, and then apply it to the design problem of an optimal servosystem. The new design method of optimal servosystems proposed here is fundamentally different from the usual ones [3),4)], and overcomes their inherent difficulties mentioned above in the following way:

(1) The design specification can be given as the time constants of first-order step responses of controlled variables.

(2) This method is computationally simple, since the primary computation required is that for pole assignment, and no Riccati solutions are required, thus leading to substantial reduction in computational time.

(3) The feedback structure of an optimal servosystem designed by this method is closely related to the system structure of the plant. In particular, realizability of state feedback by output feedback can be checked in terms of the number and values of the system zeros.

2. Problem Formulation

For a linear time-invariant, controllable and observable system

$$\begin{aligned} \dot{x} &= Ax + Bu; \quad A \in \mathbf{R}^{n\times n},\ B \in \mathbf{R}^{n\times m} \\ y &= Cx; \quad C \in \mathbf{R}^{m\times n},\ \operatorname{rank} B = m \le n \end{aligned} \tag{1}$$

we consider the design problem of optimal servosystems tracking a step reference input r. Let us denote the steady states of state x and input u for the optimal servosystem by $\bar{x}$ and $\bar{u}$, respectively and consider the following augmented system with the states $x_e = \left[(x-\bar{x})^T (u-\bar{u})^T\right]^T$, the input $v = \dot{u}$ and the output $y_e = y - r$:

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e v \\ y_e &= C_e x_e \end{aligned} \tag{2}$$

where

$$A_e = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix},\quad B_e = \begin{bmatrix} 0 \\ I \end{bmatrix},\quad C_e = [\,C \quad 0\,]$$

Then this problem can be reduced to the LQ problem for

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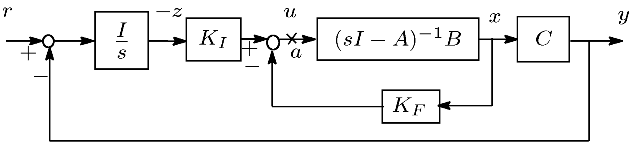

Fig. 1 General configuration of an optimal servo system

the augmented system (2) minimizing the performance cost:

$$J=\int_0^\infty \left(x_e^T Q x_e + v^T R v\right) dt;\quad Q>0,\quad R>0 \qquad (3)$$

and a solution to this problem:

$$v=-K_e x_e,\quad K_e=[K_1 \quad K_2] \qquad (4)$$

yields an optimal servosystem as in **Fig.1** (1), where

$$[\overset{n}{\widehat{K}_F} \quad \overset{m}{\widehat{K}_I}]=[\overset{n}{\widehat{K}_1} \quad \overset{m}{\widehat{K}_2}]D^{-1},\quad D\equiv\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \qquad (5)$$

We assume throughout the paper that the matrix D is nonsingular, or the system (1) has no zeros at the origin.

The objective of this paper is to provide a new LQ design theory for solving the above LQ design problem, and apply it to develop a new design method of optimal servosystems.

3. Inverse problem of optimal control

A new LQ design theory proposed here is based on some results on the inverse problem of the LQ problem (2)-(3): Given a state feedback control law (4), or a control law K_e for short, find necessary and sufficient conditions on A_e, B_e and K_e such that the control law K_e is optimal in the sense of minimizing the cost (3) for some $Q>0$ and $R>0$. First, we state the algebraic characterization of optimality of K_e as a direct consequence of LQ theory.

Lemma 1. A control law K_e is optimal if and only if there exist $P>0$ and $R>0$ satisfying

$$B_e^T P = R K_e \qquad (6)$$

and

$$P\left(\frac{1}{2}B_e K_e - A_e\right)+\left(\frac{1}{2}B_e K_e - A_e\right)^T P>0 \qquad (7)$$

In connection with this result, the following two results play important roles.

Lemma 2. The following properties hold concerning the matrices $P>0$ and $R>0$ satisfying (6). [5)]

(1) The state feedback law (5) turns out to be a solution to the LQ problem for an augmented system with the state $x_e=[\,(x-\bar{x})^T \quad z^T\,]^T$ ($\dot{z}=y_e$) and the input $v=u$, which ensures the well-known robustness at the point "a" in Fig.1.

(i) These matrices exist if and only if the following conditions hold.

A1) $K_e B_e$ has m linearly independent real left-eigenvectors, which naturally form the rows of a nonsingular real matrix V.

A2) $K_e B_e$ has m real "positive" eigenvalues, which we denote $\sigma_1,\cdots,\sigma_m$, and set $\Sigma=\mathrm{diag}\,(\sigma_1,\cdots,\sigma_m)$.

(ii) They are generally given in terms of V and Σ by

$$P=(VK_e)^T\Gamma\Sigma^{-1}(VK_e)+\begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (8)$$

$$R=V^T\Gamma V \qquad (9)$$

for some real symmetric matrices Γ and Y_1 such that

$$\Gamma>0,\quad \Gamma\Sigma=\Sigma\Gamma,\quad Y_1>0 \qquad (10)$$

Proof: See Appendix A. ∎

Lemma 3. The matrix inequality for an unknown symmetric matrix X:

$$XH+H^TX>0,\quad H\in\mathbf{R}^{(n+m)\times(n+m)} \qquad (11)$$

has a diagonal solution $X>0$ if $H=(h_{ij})$ satisfies any of the following conditions:

B1) H is copositive [6)], namely $H+H^T>0$

B2) H is column diagonal dominant [6)], namely

$$h_{ii}>c_i(H)-|h_{ii}|\equiv c_i'(H)\quad 1\le i\le n+m \qquad (12)$$

where

$$c_j(H)=\sum_{i=1}^{n+m}|h_{ij}|\quad 1\le j\le n+m$$

Proof: In the case of B1 it is obvious since $X=I$ satifies (11). The latter case of B2 is due to the reference 7). ∎

These three lemmas yield useful optimality conditions as follows:

Proposition 1. (i) Any optimal control law K_e for the LQ Problem (2)-(3) can be expressed by

$$K_e=[K_1 \quad K_2]\quad K_2=V^{-1}\Sigma V,\ K_1=K_2F \qquad (13)$$

for some real matrices V, $\Sigma>0$ and F of appropriate dimensions with V nonsingular and Σ diagonal.

(ii) For any control law K_e of the form (13), define S as a real Jordan from of $A-BF$, and T as a real nonsingular matrix such that

$$(A-BF)T=TS \qquad (14)$$

and define G, T_e and H by

$$G=-FT \qquad (15)$$

$$T_e=\begin{bmatrix} T & 0 \\ G & V^{-1} \end{bmatrix} \qquad (16)$$

$$H=T_e^{-1}\left(\frac{1}{2}B_eK_e-A_e\right)T_e \qquad (17)$$

Then the control law K_e is optimal if H is copositive or diagonal dominant.

Proof: See Appendix B. ∎

4. A new approach to LQ design

In this chapter, we develop a new solution to the LQ problem (2)-(3) based on the optimality conditions obtained in Proposition 1. The key idea is to parameterize a control law K_e in the form (13), namely,

$$K_e = V^{-1}\Sigma V\,[F \quad I] \tag{18}$$

based on the necessary condition (i), and then determine these "parameter" matrices F, V and Σ based on the sufficient condition (ii). As described above, this idea stems from the "Inverse Liner Quadratic" problem, so that we name this method "ILQ design method", and call LQ regulator obrained by this method an "ILQ regulator".

4.1 ILQ design method

As state above, the ILQ design method amounts to determining F, V and Σ so that the corresponding matrix H satisfies the condition (ii) in Proposition 1. For simplifying this process, we first set V as

$$V = I \tag{19}$$

and then determine F by pole assignment. For the V and F so determined, we finally determine Σ so that the corresponding matrix H satisfies B1 or B2. The detailed procedures are shown below.

(i) Determination of F

As stated above, we determine F by the well-established method of pole assignment with the following algorithm [8),9)]:

Step 1: Specify n stable poles $\{s_i\}$ with $s_i \neq \lambda(A)$, together with n real m-dimensional vectors $\{g_i\}$, and set

$$S = \text{(block)diag}\{s_1, \cdots s_n\} \in \mathbf{R}^{n\times n} \tag{20}$$

$$G = [g_1, \cdots, g_n] \in \mathbf{R}^{m\times n} \tag{21}$$

Step 2: Solve the linear matrix equation with an unknown $T \in \mathbf{R}^{n\times n}$:

$$AT - TS = -BG \tag{22}$$

or equivalently, determine n-dimensional vectors $\{t_i\}$ by

$$t_i = (s_i I - A)^{-1} B g_i \quad i = 1, \cdots, n \tag{23}$$

and set

$$T = [t_1, \cdots, t_n] \tag{24}$$

Step 3: Determine F by

$$F = -GT^{-1} \tag{25}$$

If T is singular, return to Step 1 and repeat the process under different choice of $\{g_i\}$.

Remark 1. This algorithm needs some modification when assigning a complex conjugate pair of poles $(s_i,\ s_{i+1})$. Namely, the sub-block $(s_i,\ s_{i+1})$ in S must be transformed into the usual real 2 x 2 block form:

$$\begin{bmatrix} a+jb & 0 \\ 0 & a-jb \end{bmatrix} \to \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \tag{26}$$

Moreover, g_i must be replaced by $g_i + jg_{i+1}$ in (23), and $(t_i,\ t_{i+1})$ in (24) by $(\mathrm{Re}\,t_i, \mathrm{Im}\,t_i)$.

Remark 2. We should note at Step 3 that the matrix T is known [8)] to be nonsingular as long as the assigned poles $\{s_i\}$ are distinct; in particular, it is always nonsingular for any choice of $\{g_i \neq 0\}$ in the single input case [10)].

(ii) Determination of Σ

For S, G, T and F as determined above, we determine $\Sigma > 0$ such that the resulting H of (17) is copositive or diagonal dominant. For this purpose, we substitute (2), (16), (18), (19) into (17) and use (14), (22) to obtain the following expression for H:

$$H = \begin{bmatrix} -S & -T^{-1}B \\ GS & \frac{1}{2}\Sigma - FB \end{bmatrix} \tag{27}$$

This expression yields the following sufficient condition for H to be copositive or diagonal dominant, which will be used later to determine Σ (see Appendix C for the proof).

Lemma 4. (i) The matrix H is copositive if

$$\mathrm{Re}\ \lambda(S) < 0 \tag{28}$$

$$\sigma_i > \underline{\sigma}_i \equiv \lambda_{\max}(E) \quad 1 \leq i \leq m \tag{29}$$

where $\lambda(S)$ denotes any eigenvalue of S, and $\lambda_{\max}(E)$ the maximum eigenvalue of the matrix:

$$E = [GS - (T^{-1}B)^T](-S - S^T)^{-1} [(GS)^T - T^{-1}B] + FB + (FB)^T$$

(ii) The matrix H is diagonal dominant if any of the following conditions holds.

(a) The matrices S and G in (20) and (21) satisfy

$$\mathrm{Re}\,s_i < 0,\ |\mathrm{Re}\,s_i| > |\mathrm{Im}\,s_i| \quad 1 \leq i \leq n \tag{30}$$

$$\begin{aligned} &\|g_i\|_1 < 0 \quad 1 \leq i \leq n \quad \text{if } s_i \in \mathbf{R} \\ &\|g_k\|_1 < \tfrac{1}{2}(1 - |\mathrm{Im}\,s_i| \,/\, |\mathrm{Re}\,s_i|) \quad k = i,\ i+1 \\ &\qquad 1 \leq i \leq n-1 \quad \text{if } s_i = \overline{s_{i+1}} \in \mathbf{C}^- \end{aligned} \tag{31}$$

where $\|\bullet\|_1$ denotes the sum of absolute values of all elements of the vector.

$$\text{(b)} \quad \sigma_i > \underline{\sigma}_i \quad 1 \leq i \leq m \tag{32}$$

where

$$\sigma_i = 2[c_i(T^{-1}B) + c_i'(FB) + (FB)_{ii}]$$

and $(FB)_{ii}$ is the $(i,\ i)$ element of FB.

(iii) Design procedure

We conclude this section with a summary of the ILQ design method developed above.

Theorem 1. By the following procedure we can design an ILQ regulator, namely a solution to LQ regulator problem (2)-(3) for some $Q > 0$ and $R > 0$.

Step 1: Choose n stable poles $\{s_i \neq \lambda(A)\}$ together with n real m-dimensional vectors $\{g_i\}$, and determine F by the preceding pole assignment algorithm.

Step 2: Choose the diagonal elements $\{\sigma_i\}$ of Σ as in (29) or as in (32), where in the latter case we must choose $\{s_i\}$ and $\{g_i\}$ so as to satisfy (30) and (31).

Step 3: Determine an optimal control law K_e by

$$K_e = \Sigma\,[F \quad I] \tag{33}$$

Remark 3. Note that the primary computation required in this method is that for pole assignment at Step 1 which is, no doubt, much simpler than that for solving Riccati equations in the usual LQ design method.

4.2 Trade-off parameters and asymptotic properties of the ILQ Regulator

As is clear from the design procedure of ILQ method described above, this method uses three kinds of design parameters $\{s_i\}$, $\{g_i\}$, and $\{\sigma_i\}$ instead of the weighting matrices Q and R as in the usual LQ design method. As for $\{\sigma_i\}$, their lower bounds are determined in Step 2, whereas no upper bounds exist. In addition, it follows from (33) that they have the function of adjusting the magnitude of optimal control inputs. In view of this observation, we use them as trade-off parameters between the magnitude of optimal control inputs and the goodness of the corresponding transient response of the ILQ regulator. The theoretical basis for the trade-off is provided by the following asymptotic property of the ILQ regulator as $\{\sigma_i\}$ tend to infinity, which has close analogy with those of the cheap LQ regulator as shown in the reference 1).

[Asymptotic property of the ILQ regulator]

Let $\{\sigma_i\}$ be of the form:

$$\sigma_i = \sigma\,\gamma_i \quad 1 \le i \le m$$

where $\{\gamma_i\}$ are any fixed positive numbers. Then as $\{\sigma_i\}$ tend to infinity in the sense of $\sigma \to \infty$, the closed-loop system of the ILQ regulator obtained in Theorem 1:

$$\dot{x}_e = F_e\,x_e \quad F_e \equiv A_e - B_e K_e \tag{34}$$

shows the following asymptotic modal property.

1) The n eigenvalues of F_e tend to $\{s_i\}$ and the corresponding eigenvectors tend to the following vectors:

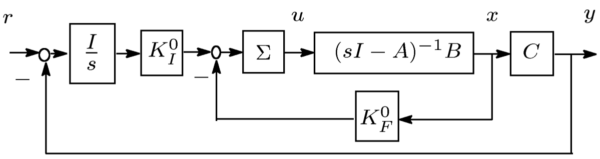

Fig. 2 General configuration of an ILQ optimal servo system

$$f_i = \begin{bmatrix} t_i \\ g_i \end{bmatrix} \quad 1 \le i \le n \tag{35}$$

2) The remaining m eigenvalues tend to $\{-\sigma_i\}$ and the corresponding eigenvectors tend to the following vectors:

$$d_i = \begin{bmatrix} 0 \\ e_i \end{bmatrix} \quad 1 \le i \le m \tag{36}$$

where $\{e_i\}$ are the natural basis of $\mathbf{R}^n$ (see Appendix D for the proof).

5. New design method of optimal servo-system

5.1 Design method

Applying the ILQ design method developed above to the optimal servo problem as stated in Chapter 2, we obtain the following basic design procedure.

[Basic design procedure]

Step 1. Specify n stable poles $\{s_i\}$ with $s_i \neq \lambda(A)$, and n real m-dimensional vectors $\{g_i\}$ as a design freedom for pole assignment

Step 2. Determine F by the pole assignment algorithm stated in Section 4.1.

Step 3. Determine $[K_F^0 \quad K_I^0]$ by

$$[K_F^0 \quad K_I^0] = [F \quad I]\,D^{-1} \tag{37}$$

Step 4. Choose m tuning parameters $\{\sigma_i\}$ as in (29) or (32).

Step 5. Configurate an optimal servosystem as in **Fig.2**, where $\Sigma = \mathrm{diag}\,(\sigma_1, \cdots, \sigma_m)$.

5.2 Choice of design parameters

The choice of design parameters $\{s_i\}$ and $\{g_i\}$ are made based on the preceding asymptotic modal property of the LQ regulator (2), (4), or equivalently the optimal servosystem of Fig.2. From this property, we first note that the output response of this servosystem has the following asymptotic behavior as $\{\sigma_i\} \to \infty$:

$$y(t) \to z(t) := r + \sum_{i=1}^{n} a_i C t_i e^{s_i t} \tag{38}$$

where $\{a_i\}$ are some scalars. Moreover, by (23), the asymptotic output response $z(t)$ is expressed concisely as

$$z(t) \ = r + \sum_{i=1}^{n} a_i W(s_i) g_i e^{s_i t} \tag{39}$$

by using the transfer function of the system (1):

$$W(s) = C(sI-A)^{-1}B \tag{40}$$

Based on this expression of $z(t)$, we choose $s_1 \sim s_n$ and $g_1 \sim g_n$ so as to yield some desirable asymptotic output response $z(t)$ of the optimal servosystem. In order to realize such a choice of $s_1 \sim s_n$ and $g_1 \sim g_n$ we first express a desired $z(t)$ in the form of modal decomposition like (39), and select them so that both responses coincide. The most simplest and practical one among them is such that each element $z_i(t)$ of $z(t)$ is expressed as a step response of a first-order system with a time constant $-1/s_i$ (>0), *i.e.*,

$$z_i(t) = r_i(1-e^{s_i t}) \qquad 1 \le i \le m \tag{41}$$

where $r = [r_1, \cdots, r_m]$. In the sequel we consider only this case for simplicity, and obtain a specific method for selecting $s_1 \sim s_n$ and $g_1 \sim g_n$.

From the definition of $z(t)$ and the restriction on $\{s_i\}$ with $s_i < 0$, The above choice is realized by choosing $s_1 \sim s_n$ and $g_1 \sim s_n$ so as to satisfy

$$\begin{array}{ll} s_i < 0 \quad Ct_i = W(s_i)g_i = \alpha_i e_i, & 1 \le i \le m \\ \text{Re } s_i < 0 \quad Ct_i = W(s_i)g_i = 0, & m+1 \le i \le n \end{array} \tag{42}$$

where $\{\alpha_i \neq 0\}$ are arbitrary real numbers. Thus the following steps realize the desired choice.

[**Selection Procedure**]

Step1. Choose any negative real numbers $s_1 \sim s_n$ that do not coincide with the poles and the zeros of the system (1).

Step2. Determine $g_1 \sim g_m$ by

$$g_i = \alpha_i W(s_i)^{-1} e_i \quad 1 \le i \le m \tag{43}$$

Step 3. Choose the remaining $s_{m+1} \sim s_n$ so as to be the zeros of the system (1) .

Step 4. Choose the remaining $g_{m+1} \sim g_n$ so as to be the associated input zero-directions.

Obviously, the necessary and sufficient conditions for feasibility of the above choice are:

C1) The sysytem (1) has the maximum number (*i.e.*, $n-m$) of zeros, namely det $CB \neq 0$.

C2) All zeros of the system (1) are stable.

We summarize the foregoing discussions as a theorem.

Theorem 2. Let us assume that the system (1) satisfies C1 and C2[(2)], and design an optimal servosystem by ILQ method with its design parameters $s_1 \sim s_n$ and $g_1 \sim g_n$ chosen by Steps 1 to 4 in the selection procedure. Then as $\{\sigma_i\} \to \infty$, each output $y_i(t)$ of the resulting optimal servo system approaches the step response of a first-order system with a time constant $-1/s_i$, *i.e.*,

$$y_i(t) \to r_i(1-e^{s_i t}) \qquad 1 \le i \le m \tag{44}$$

This result provides various practical implications which are useful in the design of optimal servosystems such as:

a) The design specifications can be given in terms of the time constant T_i of a first-order step response of each output y_i independently.

b) This specification can be achieved approximately by obvious choice of design parameters $\{s_i\}$, *i.e.*,

$$s_i = -1/T_i \quad 1 \le i \le m$$

and hence the decoupling control of the outputs can be achieved approximately.

c) The tracking performance of the optimal servo system can be adjusted by tuning $\{\sigma_i\}$, while compromising the magnitude of control inputs.

Remark 4 If we choose $\{\sigma_i\}$ based on (32) at Step 4 of the design procedure in Section 5.1, we need to adjust the magnitude of $g_1 \sim g_n$ so as to satisfy (31). Moreover, it is better to increase the magnitude of each g_i as much as possible, so that $\underline{\sigma}_i$ becomes small. The system zeros and zero-directions required in Steps 3 and 4 can be obtained by solving (42), or equivalently the associated generalized eigenvalue problem [11)]:

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} t_i \\ g_i \end{bmatrix} = s_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t_i \\ g_i \end{bmatrix} \qquad m+1 \le i \le n \tag{45}$$

In regard to the nonsingularity of T as required in the pole assignment of Step 2 in connection with the choice of $\{s_i\}$ and $\{g_i\}$ at Step 1 to 4, we should note the statement made in Remark 2.

5.3 Feedback structure

The optimal servo system of Fig.2 obtained in Theorem 2 shows a special structure of its feedback matrices K_F^0 and K_I^0.

Theorem 3. Under the assumptions of C1 and C2, the optimal servosystem obtained in Theorem 2 can be realized in the form of output feedback configuration as shown in **Fig.3**, where the feedback matrices are given by

$$\begin{aligned} K_y^0 &= (CB)^{-1} \\ K_I^0 &= -K_y^0 \text{diag}(s_1, \cdots, s_m) \end{aligned} \tag{46}$$

Proof. By post-multiplying the matrix DT_e with $V = I$ on both sides of (37), and using (22) and (25), we see that K_F^0 and K_I^0 are uniquely determined by solving

$$[K_F^0 \quad K_I^0] \begin{bmatrix} TS & B \\ CT & 0 \end{bmatrix} = [0 \quad I] \tag{47}$$

(2) These two condisions are void in the case of $n = m$

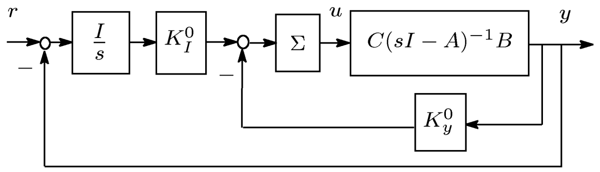

Fig. 3 Configuration of an ILQ optimal servo system by output feedback

Moreover, we see from (42) that this is satisfied by $K_F^0 = K_y^0 C$ and the K_I^0 given by (46) with $K_y^0 = (CB)^{-1}$. ∎

Remark 5. As shown above, the state feedback of the optimal servosystem can be realized by a suitable output feedback. In addition, it follows easily from (47) that, when we choose $g_1 \sim g_n$ by (43) at Step 1 of design procedure, the converse statement is also true, and the corresponding K_F^o, K_I^o are given by (46).

6. Design example

In this section we illustrate a design example of the optimal servosystem obtained in Theorem 2. Consider the system (1) with the following coefficient matrices and the transfer function matrix:

$$A = \begin{bmatrix} -0.4 & -1 & 0 \\ 0 & -8 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$W(s) = \frac{1}{s(s+8)(s+0.4)} \begin{bmatrix} s(s+8) & s+2 \\ 0 & -(s+0.4)(s+2) \end{bmatrix}$$

Note that this system satisfies the conditions C1 and C2 in Theorem 2, and has a zero of -2.

Step 1. Specify the time constants T_1, T_2 of a first-order step response of the outputs y_1, y_2, respectively, and choose $s_1 \sim s_3$ as

$$s_1 = -\frac{1}{T_1}, \quad s_2 = -\frac{1}{T_2}, \quad s_3 = -2$$

Accordingly, choose g_1, g_2 by (43), i.e.,

$$g_1 = \alpha_1 W(s_1)^{-1} e_1 = \alpha_1 [s + 0.4 \quad 0]^T$$
$$g_2 = \alpha_2 W(s_2)^{-1} e_2 = \alpha_2 [1 \quad -s_2(s_2+8)/(s_2+2)]^T$$

and g_3 by the latter equation of (42), namely, choose it as a solution of $W(s_3)g_3 = 0$ as follows:

$$g_3 = \alpha_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Here we determine each α_i such that $\|g_i\|_1 = 1$.

Step 2. First compute $t_1 \sim t_3$ by (23) as

$$t_1 = \alpha_1 [1 \quad 0 \quad 0]^T$$
$$t_2 = \alpha_2 [0 \quad 1 \quad -2(s_2+8)/(s_2+2)]^T$$
$$t_3 = \alpha_3 [0 \quad 0 \quad -1]^T$$

and F by (25) as

$$F = \begin{bmatrix} -s_1 - 0.4 & -1 & 0 \\ 0 & s_2 + 8 & 1 \end{bmatrix}$$

Step 3. Compute the gain matrices K_F^0, K_I^0 of Fig.2 as

$$[K_F^0 \quad K_I^0] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -s_1 & 0 \\ 0 & -1 & 0 & 0 & s_2 \end{array}\right]$$

Note that this result can be obtained directly from Theorem 3.

Step 4. Choose $\{\sigma_i\}$ by (29) or (32).

As an example, we shall show the simulation result for the case of $T_1 = T_2 = 1$. First, the conditions (29), (32) for selection of the tuning parameters are as follows:

$$\sigma_1 > 1.14, \quad i = 1, 2 \quad \text{(coopositivity)} \qquad (48)$$

$$\sigma_1 > 2.4, \quad \sigma_2 > 32 \quad \text{(diagonal dominance)} \qquad (49)$$

By noting (49) we fix σ_2 as $\sigma_2 = 33$, and change σ_1, for which the resulting input and output responses of the control system of Fig.2 are shown in **Fig.4**. Obviously (a) means that the asymptotic property (44) for the output y_1 as established in Theorem 2 holds true; (b) shows that the response of y_2 has almost reached to a desired first-order response since σ_2 is large enough; (c) indicates that the response of u_1 becomes fast as σ_1 increases. Hence the value of σ_1 should be determined by trade-off between the tracking performance of y_1 and the speed of the response of u_1. Similarly, we determine the value of σ_2 based on the responses of y_2 and u_2 , which are omitted. As is clear from Fig. 4, the tracking performance of outputs hardly change for any values of $\{\sigma_i\}$ larger than a certain level. However, the input responses become fast at any rate of speed in accordance with the increase of σ_1 . Therefore, it is not only practical to increase $\{\sigma_i\}$, but also inappropriate from the viewpoint of robust stability; in fact, a slight variation of B sometimes results in the instability of the closed-loop system in Fig.3, and hence we should be careful about it.

7. Conclusion

By taking a new look at LQ theory from the viewpoint of the inverse LQ problem, we have obtained a practical LQ design theory and applied it to the design of optimal servosystems. Although our attention was restricted to the particular system, the results presented here can be easily extended to general linear time-invariant systems.

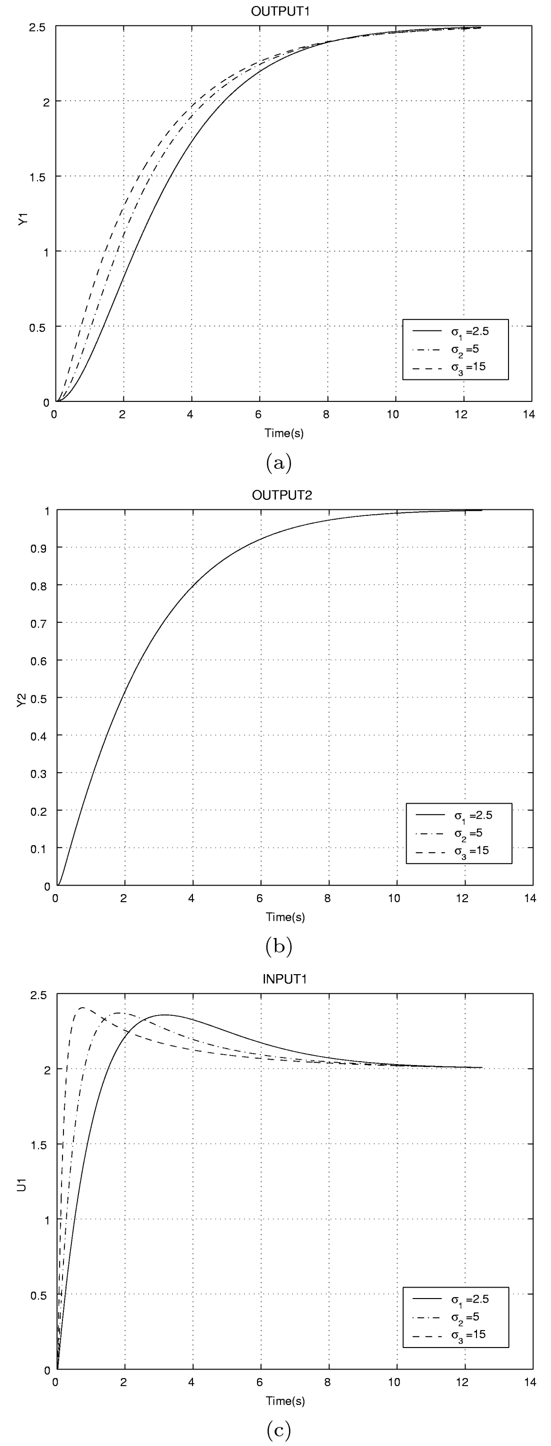

Fig. 4 Step responses of an ILQ optimal servo system for various values of σ_1 with σ_2 fixed

Similarly, the proposed selection method of design parameters, though it has a certain restricted application, can also be extended to the one without restrictions. These matters will be discussed in the future paper, and thus the discussion was focused here on its basic one.

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Appendix A.

Noting rank $B_e = m$ by definition, we apply some results of the reference 5) to the algebraic equation (6). First, the necessity of (i) is obvious from Theorem 2.2 in the reference 5), and so is it from Lemma 2.1 (ii) and Theorems 3.1, 3.4, 4.1 and the definition of B_e that an arbitrary set of solutions $P > 0, R > 0$ of (6) are expressed by (8), (9). The converse fact follows easily by applying Theorems 3.1, 4.1 and noting the following relation as a direct consequence of A2:

$$\text{rank } K_e B_e = \text{rank } K_e = \text{rank } B_e \qquad (\text{A. }1)$$

This completes the proof for sufficiency of (i) as well as that of (ii).

Appendix B.

(i) By definition $K_2 = K_e B_e$, which satisfies A1 and

A2 by Lemma 1 and Lemma 2(i) if K_e is optimal. Hence with V, Σ defined in Lemma 2 (i), the first relation of (13) is obvious. The second relation is also obvious with $F = K_2^{-1}K_1$, since K_2 is nonsingular.

(ii) Since $K_eB_e = K_2 = V^{-1}\Sigma V$ implies the conditions A1 and A2 in Lemma 2, it suffices by Lemma 1 and Lemma 2 (ii) to show that there exists a solution $P > 0$ to (6) satisfying (7) under the conditions B1 or B2. In other words, it suffices to show under the above conditions that, for some solution $P > 0$ of (6) and the H of (17), the matrix

$$X = T_e^T P T_e \tag{B.1}$$

satisfies (11). To show this, substituting (16), (8) into this, and noting (13), (15) yield

$$X = \mathrm{diag}(T^T Y_1 T, \Sigma\Gamma), \quad Y_1 > 0, \quad \Gamma > 0 \tag{B.2}$$

This obviously means that X could be an arbitrary positive definite diagonal matrix by proper choice of $Y_1 > 0$ and $\Gamma > 0$, or a suitable solution $P > 0$ of (6), and therefore X could be a positive definite diagonal solution of (11) by Lemma 3.

Appendix C.

(i) From the form of H in (27) and a well known property related to the positive definiteness of a symmetric matrix, $H + H^T < 0$ if and only if

$$S + S^T < 0, \quad \Sigma - E > 0 \tag{C.1}$$

By the structure of S in (20), the first half is equivalent to (28), and the second half holds if

$$\sigma_{min} I > E \quad (\sigma_{min} \equiv \min\{\sigma_i\}) \tag{C.2}$$

namely, (29) holds, since E is real and symmetric.

(ii) From the structures of H in (27) and S in (20), it follows easily that the conditions (30), (31) give a sufficient condition for column diagonal dominance with respect to the first n columns of H, and the condition (32) gives the one with respect to the remaining m columns.

Appendix D.

Note from (18) and the special form of σ_i that $K_e = \sigma K_f [K_s \quad I]$ with $K_f = V^{-1}\mathrm{diag}(\gamma_1, \cdots, \gamma_m)V$ and $K_s = F$, and note also that the eigenvalues of $A - BF$ are $\{s_i\}$ and those of K_f are $\{\gamma_i\}$. Then the asymptotic eigenvalue properties as stated in 1) and 2) follow directly from Theorem 2 of the reference 12). To show the asymptotic eigenvector property for F_e, we first consider $F \equiv T_e^{-1} F_e T_e$, which is expressed by (17), (27) and (18), (19) as

$$F = \begin{bmatrix} S & T^{-1}B \\ -GS & FB - \sigma\Gamma \end{bmatrix}, \quad \Gamma \equiv \mathrm{diag}(\gamma_i, \cdots, \gamma_m)$$

Denote the eigenvalue of F by λ_i and the corresponding eigenvector by

$$z_i = \begin{bmatrix} z_{i1} \\ z_{i2} \end{bmatrix} \begin{matrix} \}n \\ \}m \end{matrix}$$

Then we have $Fz_i = \lambda_i z_i, i.e.,$

$$(S - \lambda_i I)z_{i1} + T^{-1}Bz_{i2} = 0$$
$$-GSz_{i1} + (FB - \sigma\Gamma - \lambda_i I)z_{i2} = 0$$

Using this relation and the preceding asymptotic eigenvalue properties, we easily see that as $\sigma \to \infty$,

1) $z_{i1} \to p_i$ and $z_{i2} \to 0$ if $\lambda_i = s_i \quad 1 \le i \le m$

2) $z_{i2} \to 0$ and $z_{i2} \to e_i$ if $\lambda_i = -\sigma\gamma_i \quad m+1 \le i \le n$

where p_i is the i th eigenvector of S associated with an eigenvalue s_i of S. Transforming this eigenvector property of F into that of F_e yields the desired result.

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