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Structural Analysis of Fault-Tolerance for Homogeneous Systems†

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This paper investigates fault-tolerance of homogeneous systems that consist of a number of identical subsystems. In order to retain the fault-tolerance of large-scale systems, not only the reliability of each component, but also the design of a whole system counts for much. We carried out quantitative evaluation of fault-tolerance in terms of autonomous controllability for systems with different structures. The failure patterns to cause the systems to be uncontrollable can be found merely by structural information. In consequence of this investigation, we focused our attention on symmetrical structures of the systems, and applied group representation theory to our analysis.

Key Words: large-scale systems, interconnection, fault-tolerance, controllability, group representation.

1. Introduction

A variety of ideas have been proposed concerning how to construct and how to manage large-scale systems, as their size and complexity have shown a rapid increase. The main point to construct a large-scale system is to unify a number of small components through the interconnections between them. That is why researches have been carried out for hierarchical multilevel systems [1)] and autonomous decentralized ones [2)]. In addition, the lack of reliability may cause serious problems especially for large-scale systems, because failures may occur more frequently, or, it may not be possible to react immediately against the failures, for example in space stations, intelligent vehicle systems, or, distributed control systems of factory-automation [3)]. In order to retain the fault-tolerance of large-scale systems, not only the reliability of each component, but also the design of a whole system counts for much. Therefore, it is useful to carry out structural analysis of fault-tolerance for large-scale systems.

Recently, reliability of systems have been increasingly studied. For example, 4) and 5) considered passive redundant controllers, and integrity conditions have been discussed in 6)−9). A considerable number of the fault-tolerant systems are also reported from the practical fields (for example, see 11)). These reports are accompanied by the studies on protocols for exchange of information between subsystems. However, few researches have been conducted on the physical connections among the subsystems. Then the problems come about: "How should the subsystems be interconnected one another to attain fault-tolerance as a whole?" This paper carries out quantitative analysis for fault-tolerance of the systems with different structures aiming at pursuing "good" interconnections.

As a measure of fault-tolerance, autonomous controllability of the entire system [12)] has been defined for multivariable systems. This property guarantees the controllability of the entire system even with the failures in some control channels. Moreover, the definition of autonomous controllability is accompanied by the number of control channels that have failed, so that this number should serve for a quantitative evaluation of fault-tolerance. If a system remains controllable even with m channels in the outage, this system is said to be autonomously controllable at level m. This is the similar idea with m-actuator-integrity by Gündeş [8)], in the sense that it could adopt the acceptable number of failures to keep certain properties. Furthermore, a design method has been proposed [13)] for such systems that remain stable against certain number of failures, i.e., systems with integrity at a certain level of failures.

We investigate the characteristics determined merely by the structure of interconnections. Hence, all components and the existing interconnections among them are supposed to be identical, and such systems are called *homogeneous systems*. From the practical view, assemblage of identical units has the advantage to facilitate analysis and

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design of the entire systems, and enables to get a variety of systems. Replacements of the components that have failed could be facilitated by their mass-production. It is therefore suitable to the recent demands on the production of small quantity with more variety. Some examples are seen in robotic systems [14]. Especially, we shall discuss the following three kinds of basic structures (Fig. 1): (a) ring-type, (b) chain-type, and (c) wheel-type homogeneous systems.

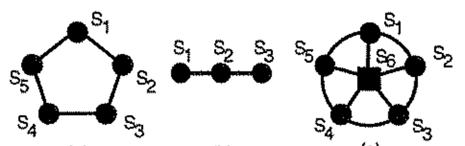

Fig. 1 Homogeneous systems considered in this paper. (a) Ring-type, (b) chain-type, (c) wheel-type systems.

We carry out structural analysis of fault-tolerance for such homogeneous systems and show the failure patterns that cause the systems to be uncontrollable. At last, we mentioned symmetric homogeneous systems defined by the use of structural symmetry [16], and show that analysis of fault-tolerance can profit from group representation theory [15].

This paper is organized as follows. Section 2 introduces the representation of structured homogeneous systems considered in this paper, and a measure of fault-tolerance utilizing autonomous controllability of the entire systems. In Section 3, we show the procedure to analyze the failures that cause the homogeneous systems to be generically uncontrollable and give the results for three homogeneous systems. Section 4 considers application of group theory to the analysis discussed in Section 3.

In this paper, I_k denotes the unit matrix of order k, and $R(A,B)$ denotes the controllability matrix of (A,B), i.e., $R(A,B) = [B\ AB\ A^2B \cdots A^{n-1}B]$.

2. System structure and a measure of fault-tolerance

2.1 Homogeneous systems

Consider a system S obtained by connecting m subsystems $\{S_1, S_2, \cdots, S_m\}$. Each subsystem S_i (i=1, 2, $\cdots$, m) is described by a state transition equation

$$\dot{x}_i(t) = A_{ii}x_i(t) + B_{ii}u_i(t) + \sum_{j\neq i} A_{ij}x_j(t) + \sum_{j\neq i} B_{ij}u_j(t),$$

where $x_i(t) \in \mathbf{R}^{n_i}$ and $u_i(t) \in \mathbf{R}^{r_i}$ denote the state of S_i and the input from its control channel, respectively. The entire system S is described in the standard form of a state transition equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix},$$

where the state and the input of the entire system S are denoted as $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^r$, respectively. Matrices A_{ij} and B_{ij} are of consistent dimensions. The diagonal blocks of A and B stand for the effects of each subsystem S_i on its state x_i. On the other hand, the off-diagonal blocks imply the interactions among subsystems.

If all the m subsystems S_i (i=1, 2, $\cdots$, m) are identical, i.e., $n_i = n_0$, $r_i = r_0$, $A_{ii} = K$, $B_{ii} = P$,

$$A_{ij} = \begin{cases} L & \text{(if } S_i \text{ and } S_j \text{ are connected)} \\ 0 & \text{(otherwise)}, \end{cases}$$

$$B_{ij} = \begin{cases} Q & \text{(if } S_i \text{ and } S_j \text{ are connected)} \\ 0 & \text{(otherwise)}, \end{cases}$$

where K, L, P and Q are nonzero matrices with consistent dimensions, S is called a *homogeneous system*.

In the following we are interested in the characteristics defined by its structure. Therefore, the numerical information about K, L, P and Q is not required. The structured system defined with these matrix parameters is denoted by $(\bar{A}, \bar{B})$. For example, $(\bar{A}, \bar{B})$ of each homogeneous system considered here (Fig. 1) is described as follows:

(a) Ring-type

$$\left(\begin{bmatrix} K & L & O & \cdots & O & L \\ L & K & \ddots & & & O \\ O & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & O \\ O & & & \ddots & K & L \\ L & O & \cdots & O & L & K \end{bmatrix}, \begin{bmatrix} P & Q & O & \cdots & O & Q \\ Q & P & \ddots & & & O \\ O & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & O \\ O & & & \ddots & P & Q \\ Q & O & \cdots & O & Q & P \end{bmatrix} \right),$$

where the index number of subsystems is assigned clockwise from an arbitrary subsystem.

(b) String-type

$$\left(\begin{bmatrix} K & L & O & \cdots & O \\ L & K & \ddots & \ddots & \vdots \\ O & \ddots & \ddots & \ddots & O \\ \vdots & \ddots & \ddots & \ddots & L \\ O & \cdots & O & L & K \end{bmatrix}, \begin{bmatrix} P & Q & O & \cdots & O \\ Q & P & \ddots & \ddots & \vdots \\ O & \ddots & \ddots & \ddots & O \\ \vdots & \ddots & \ddots & \ddots & Q \\ O & \cdots & O & Q & P \end{bmatrix} \right),$$

where the subsystem of one side is assigned to S_1, from which the index number is assigned in regular order to the subsystem S_m of the other side.

(c) Wheel-type

$$\left(\begin{bmatrix} K & L & O & \cdots & O & L & L \\ L & K & L & \ddots & & O & L \\ O & L & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & O & L \\ O & & \ddots & \ddots & \ddots & L & L \\ L & O & \cdots & O & L & K & L \\ L & L & \cdots & L & L & L & K \end{bmatrix}, \begin{bmatrix} P & Q & O & \cdots & O & Q & Q \\ Q & P & Q & \ddots & & O & Q \\ O & Q & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & O & Q \\ O & & \ddots & \ddots & \ddots & P & Q \\ Q & O & \cdots & O & Q & P & Q \\ Q & Q & \cdots & Q & Q & Q & P \end{bmatrix} \right),$$

where the subsystem at the center is S_m and the index number one to $m-1$ of subsystems is assigned clockwise from an arbitrary subsystem.

All the information about structure of the systems is sufficiently described in a matrix Δ of order m obtained by replacing K and L in $\bar{A}$ with scalar parameters. Moreover, the structured systems $(\bar{A}, \bar{B})$ is said to be generically uncontrollable if it is always uncontrollable in spite of the numerical information about K, L, P and Q.

2.2 A measure of fault-tolerance

A measure of fault-tolerance is defined as follows, by adopting autonomous controllability of the entire system [12] to "cut" in the study of the reliability of systems [17].

We first associate with each subsystem S_i a variable f_i such that

$$f_i = \begin{cases} 0 \text{ (if } S_i \text{ is faulty),} \\ 1 \text{ (otherwise).} \end{cases} \quad (1)$$

In this paper, failure of a subsystem is restricted to that of its control channel [2]. In other words, to say a failure occured in a subsystem S_i at time t_0 means

$$u_i(t) = 0 \qquad \text{for} \quad \forall t \geq t_0. \quad (2)$$

Systems for practical use may be equipped to make the controller stop when any kind of doubtful signals are detected. Then, the suspected parts should be inspected with their control inputs off. Therefore, this definition of failures as (2) can deal with the case of emergency stops, or, maintenance of subsystems. Even in such cases, fault-tolerant systems are desired to keep functioning as a whole.

We especially consider the controllability of the systems with some failures. First of all, systems without failures in any subsystems are supposed to be controllable. The information we have is only the connections among the subsystems described in $(\bar{A}, \bar{B})$, and the variables f_i in (2). Therefore, for each $(\bar{A}, \bar{B})$, the entire system becomes generically uncontrollable depending on $(f_1, f_2, \cdots, f_m)$. Let $\bar{S}$ be the subset of subsystems whose indices belong to J. If the entire system is generically uncontrollable when all the subsystems in $\bar{S}$ have failed, i.e., $f_j = 0$ for $j \in J$ and $f_j = 1$ for $j \notin J$, one says that $\bar{S}$ is a *cut*. Then the number of the subsystems in a cut may manifest the level of fault-tolerance of the homogeneous systems to be generically controllable.

A failure matrix is then defined by $F = \text{diag}\{f_1, f_2, \cdots, f_m\}$, corresponding to a pattern of failures. The matrix $\bar{B}F_{r_0}$, where $F_{r_0} = F \otimes I_{r_0}$, has zero column blocks that correspond to all the subsystems in the outage.

Remark.

To consider the controllability of the entire system after the failures in some subsystems is useful for the systems in which total replacements of the subsystems are impossible, for example, in space station systems [18] and large-scale factory automation systems [3]. On the other hand, autonomous controllability defined by Mori *et al.* [2] is useful if it is possible to remove the subsystems in the outage, for example, in computer network systems [10]. Their autonomous controllability is the controllability of the states of the functioning subsystems with all the failures in some subsystems, so that it makes no mention of the states of subsystems in the outage.

3. Evaluation of fault-tolerance for homogeneous systems

To evaluate fault-tolerance of structured homogeneous systems, we investigate the cut $\bar{S}$ described above, where the generic rank of $(\bar{A}, \bar{B}F_{r_0})$ becomes less than n. An analysis procedure is described as follows.

Find a matrix Z such that $Z^{-1}\Delta Z$ is diagonal. Then $\tilde{A} = (Z \otimes I_{n_0})^{-1}\bar{A}(Z \otimes I_{n_0})$ becomes block diagonal, say, block-diag$\{A_1, A_2, \cdots, A_m\}$. Because the rank of the controllability matrix $R(\bar{A}, \bar{B})$ is invariant under state transformations, we shall deal with $R(\tilde{A}, \tilde{B})$, where $\tilde{B} = (Z \otimes I_{n_0})^{-1}\bar{B}$. Since $\tilde{B}(Z \otimes I_{r_0})$ is also block-diagonal, say, block-diag$\{B_1, B_2, \cdots, B_m\}$, we can get

$$\tilde{A}^i\tilde{B} = \text{diag}\{A_1^iB_1\ A_2^iB_2 \cdots A_m^iB_i\}(Z \otimes I_{r_0})^{-1}.$$

By permutating some rows and columns,

$$\text{rank}R(\bar{A}, \bar{B}F_{r_0}) = \text{rank}\left\{\begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_m \end{bmatrix} Z^{-1}F \otimes I_{nr_0}\right\},$$

where $R_j = \begin{bmatrix} B_j\ A_jB_j\ A_j^2B_j \cdots A_j^{n-1}B_j \end{bmatrix}$ $(j = 1, \cdots, m)$. Here, we suppose that R_j is of full-rank, i.e., $R(A_j, B_j)$ is of full-rank because the order of A_j is n_0. Under this condition, uncontrollability after failures can occur either by the existence of a zero row in $Z^{-1}F$, or, the dependency of rows in $Z^{-1}F$ corresponding to the identical blocks in $\tilde{A}$.

The discussion above leads us to the following procedure for analysis on autonomous controllability of $(\bar{A}, \bar{B})$.

(1) Find a constant matrix Z which makes $\tilde{\Delta} = Z^{-1}\Delta Z$ diagonal, independently of the parameters in Δ.

(2) Examine the zero and nonzero pattern of Z^{-1}.

(3) Examine the dependence of rows in Z^{-1}, corresponding to the identical blocks of $\tilde{A}$.

We follow this procedure for three basic homogeneous systems.

(a) The matrix Z for ring-type systems is obtained as follows by a group theoretic approach. This procedure shall be explained in the next section.

If m is an odd number, Z is obtained as

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\theta & \sin\theta & \cdots & \cos\lfloor\frac{m-1}{2}\rfloor\theta & \sin\lfloor\frac{m-1}{2}\rfloor\theta \\ \frac{1}{\sqrt{2}} & \cos 2\theta & \sin 2\theta & \cdots & \cos 2\lfloor\frac{m-1}{2}\rfloor\theta & \sin 2\lfloor\frac{m-1}{2}\rfloor\theta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{2}} & \cos m\theta & \sin m\theta & \cdots & \cos m\lfloor\frac{m-1}{2}\rfloor\theta & \sin m\lfloor\frac{m-1}{2}\rfloor\theta \end{bmatrix}, \quad (3)$$

and if m is an even number, Z is obtained as

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\theta & \sin\theta & \cdots & \cos\lfloor\frac{m-1}{2}\rfloor\theta & \sin\lfloor\frac{m-1}{2}\rfloor\theta & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \cos 2\theta & \sin 2\theta & \cdots & \cos 2\lfloor\frac{m-1}{2}\rfloor\theta & \sin 2\lfloor\frac{m-1}{2}\rfloor\theta & -\frac{1}{\sqrt{2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{2}} & \cos m\theta & \sin m\theta & \cdots & \cos m\lfloor\frac{m-1}{2}\rfloor\theta & \sin m\lfloor\frac{m-1}{2}\rfloor\theta & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad (4)$$

where $\theta = \frac{2\pi}{m}$. For both cases, Z^{-1} is equal to $\frac{2}{m}Z^{\mathrm{T}}$.

(b) The matrix Z for chain-type systems is explicitly obtained for tridiagonal matrices as follows.

$$Z = \begin{bmatrix} \sin\theta & \sin 2\theta & \cdots & \sin m\theta \\ \sin 2\theta & \sin 4\theta & \cdots & \sin 2m\theta \\ \vdots & \vdots & \vdots & \vdots \\ \sin m\theta & \sin 2m\theta & \cdots & \sin m^2\theta \end{bmatrix}, \quad (5)$$

where $\theta = \frac{\pi}{m+1}$ and $Z^{-1} = \frac{2}{m+1}Z^{\mathrm{T}}$.

(c) The matrix Z for wheel-type systems is calculated as follows by the aid of Z for ring-type systems.

If $m' = m-1$ is an odd number, Z is obtained as

$$\begin{bmatrix} \cos\theta & \sin\theta & \cdots & \cos\lfloor\frac{m'-1}{2}\rfloor\theta & \sin\lfloor\frac{m'-1}{2}\rfloor\theta & 1/2 & 1/2 \\ \cos 2\theta & \sin 2\theta & \cdots & \cos 2\lfloor\frac{m'-1}{2}\rfloor\theta & \sin 2\lfloor\frac{m'-1}{2}\rfloor\theta & 1/2 & 1/2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos m'\theta & \sin m'\theta & \cdots & \cos m'\lfloor\frac{m'-1}{2}\rfloor\theta & \sin m'\lfloor\frac{m'-1}{2}\rfloor\theta & 1/2 & 1/2 \\ 0 & 0 & \cdots & 0 & 0 & \frac{\sqrt{m'}}{2} & -\frac{\sqrt{m'}}{2} \end{bmatrix},$$

and if $m' = m-1$ is an even number, Z is obtained as

$$\begin{bmatrix} \cos\theta & \sin\theta & \cdots & \cos\lfloor\frac{m'-1}{2}\rfloor\theta & \sin\lfloor\frac{m'-1}{2}\rfloor\theta & \frac{1}{\sqrt{2}} & 1/2 & 1/2 \\ \cos 2\theta & \sin 2\theta & \cdots & \cos 2\lfloor\frac{m'-1}{2}\rfloor\theta & \sin 2\lfloor\frac{m'-1}{2}\rfloor\theta & -\frac{1}{\sqrt{2}} & 1/2 & 1/2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos m'\theta & \sin m'\theta & \cdots & \cos m'\lfloor\frac{m'-1}{2}\rfloor\theta & \sin m'\lfloor\frac{m'-1}{2}\rfloor\theta & -\frac{1}{\sqrt{2}} & 1/2 & 1/2 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \frac{\sqrt{m'}}{2} & -\frac{\sqrt{m'}}{2} \end{bmatrix}$$

where $\theta = \frac{2\pi}{m'}$. For both cases, Z^{-1} is calculated as $\frac{2}{m-1}Z^{\mathrm{T}}$.

For each of the matrix Z obtained above, we examine the zero and nonzero pattern of Z^{-1}. For example, the (i,j)–component $(1 \le i,j \le m)$ of (5) described as $\sin(ij\theta)$ is zero if and only if $ij \equiv 0 \pmod{m+1}$. The discussion above, together with the following elementary lemma, leads us to the following theorem. Similar analysis can be carried out for ring-type and wheel-type systems.

Lemma 1. Suppose that $1 \le i \le m$ $(m > 1)$, there exists an integer j which satisfies

$$1 \le j \le m \quad \text{and} \quad ij \equiv 0 \pmod{m+1},$$

if and only if i is a divisor of $m+1$ with $i > 1$.

Theorem 1.

The cuts of three basic structures to be generically uncontrollable are described as follows.

(a) The cuts of the ring-type structure are $\mathcal{S} - \{\mathcal{S}_q, \mathcal{S}_{2q}, \mathcal{S}_{3q}, \cdots, \mathcal{S}_m\}$, for all the divisors q of m, except for the case $m \equiv 2 \pmod 4$ where $q = 2$ is not included.

(b) The cuts of the chain-type structure are $\mathcal{S} - \{\mathcal{S}_q, \mathcal{S}_{2q}, \mathcal{S}_{3q}, \cdots, \mathcal{S}_{m+1-q}\}$, for all the divisors $q > 1$ of $(m+1)$.

(c) The cuts of the wheel-type structure are $\mathcal{S} - \{\mathcal{S}_q, \mathcal{S}_{2q}, \cdots, \mathcal{S}_{m-1}, \mathcal{S}_m\}$, for all the divisors q of $(m-1)$, except for the case $(m-1) \equiv 2 \pmod 4$ where $q = 2$ is not included.

Example.

Figure 2 gives examples of generically uncontrollable systems according to the theorem shown above. Failures occur in the subsystems represented by dotted circles.

In the failure patterns that induce uncontrollability as shown in Theorem 1, the functioning subsystems are distributed in periodic ways (see Fig. 2). This periodicity derives from two kinds of symmetry. Namely, if a system has structural symmetry, the system is not tolerant for failures of subsystems distributed symmetrically. In this context, it might be worth mentioning that ring-type homogeneous systems with a prime number m, chain-type homogeneous systems with a prime number $m+1$, and wheel-type homogeneous systems with a prime number $m-1$ are superior in fault tolerance. In these cases, the system never becomes uncontrollable by symmetry as far as arbitrary two subsystems remain functioning.

Example. As another examples, consider the systems shown in Fig. 3 (a) and (c). The minimum number of subsystems to cause generic uncontrollability in each system (a), (b), and (c) in Fig. 3 is then derived as 2, 4, and 4, respectively. This result suggests that the system with

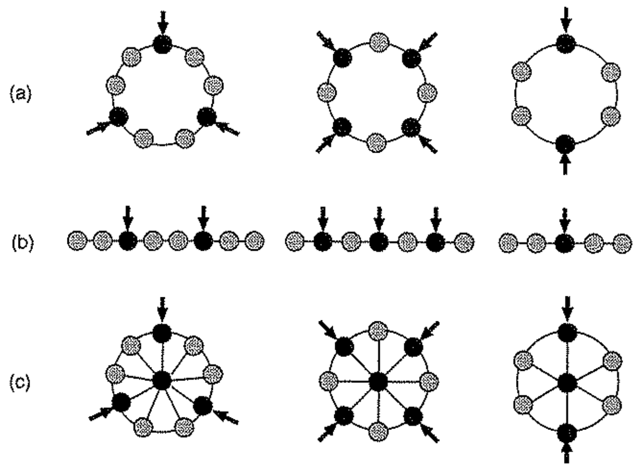

Fig. 2 Examples of the homogeneous systems which are generically uncontrollable. Failures occur in the subsystems represented by dotted circles. (a) Ring-type, (b) chain-type, (c) wheel-type homogeneous structured systems.

structure (a) is less fault-tolerant than the others in a sense. From neuroethology, it is known that the nerves in the six legged insects adopt (c) type interconnection [19].

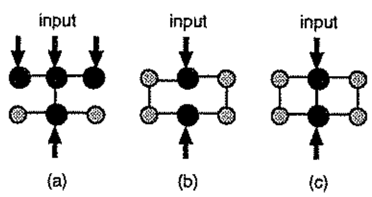

Fig. 3 Homogeneous systems with six subsystems. Failures occur in the subsystems represented by dotted circles. We can see these systems are generically uncontrollable. It is said that the interconnections among the six legs of insects adopt (c) style.

4. Application of group representation

During the procedure shown in the previous section, the primary point is the diagonalization of Δ. This section presents a group theoretic background for finding the matrix Z for systems with symmetrical structures.

Before moving on to the definition of symmetric systems, we introduce some fundamental facts about group representations (see, for example 15)) utilized in the following discussion.

Let G be a finite group. Let V be a finite-dimensional vector space over $\mathbf{K}$ $=\mathbf{R}$ or $\mathbf{C}$, where $\mathbf{R}$ is the field of real numbers and $\mathbf{C}$ is the field of complex numbers, and denote by $GL(V)$ the group of all nonsingular linear transformations of V onto itself. A *representation* of G on *representation space* V is a homomorphism $\tau : G \to GL(V)$. The *dimension* of the representation is $n = \dim V$. A subspace W of V is *invariant* under τ if $\tau(g)w \in W$ for every $g \in G, w \in W$. The representation τ is *irreducible* if the only invariant subspaces of V are $\{0\}$ and V itself. An n–dimensional matrix representation of G is a homomorphism $T : G \to GL(n, \mathbf{K})$, where $GL(n, \mathbf{K})$ denotes the group of all nonsingular matrices over $\mathbf{K}$ of order n. If a basis $\{v_1, v_2, \cdots, v_n\}$ is fixed for V, we obtain a matrix representation T of G. The character $\chi : G \to \mathbf{K}$ of τ is defined by $\chi(g) = \mathrm{Tr}\tau(g) = \mathrm{Tr}T(g)$. Note that the character χ is independent of the choice of basis vectors for V.

We denote by $\{\tau^\mu \mid \mu \in R(G)\}$ a complete list of nonequivalent irreducible representations of G, where $R(G)$ denotes an index set for the irreducible representations of G. A complete list of nonequivalent irreducible matrix representations of G is denoted by $\{T^\mu \mid \mu \in R(G)\}$. We denote the dimension of T^μ by n^μ. Every finite-dimensional representation of a finite group can be decomposed into a direct sum of irreducible representations. The direct sum decomposition is obtained by

$$V = \oplus_{\mu \in R(G)} \oplus_{i=1}^{a^\mu} V_i^\mu,$$

where V_i^μ are invariant subspaces of V which transform irreducibly under the restrictions τ^μ of τ to V_i^μ, and the *multiplicity* a^μ of τ^μ in τ is uniquely determined. Then the matrix representation can be put into a block-diagonal form

$$T(g) = \oplus_{\mu \in R(G)} \oplus_{i=1}^{a^\mu} T_i^\mu(g), \qquad g \in G,$$

where T_i^μ is irreducible, if we first choose a basis $\{v_{ij}^\mu | j = 1, \cdots, n^\mu\}$ for each V_i^μ and adopt their union as a basis of V. Moreover, by choosing a basis adequately, we can have $T_i^\mu(g) = T^\mu(g)$ $(1 \le i \le a^\mu)$. Namely, the decomposition is as

$$T(g) = \oplus_{\mu \in R(G)} \oplus_{i=1}^{a^\mu} T^\mu(g). \tag{6}$$

The following is known as Schur's lemma, see, e.g., 15).

Lemma 2.
Let T_1 and T_2 be irreducible matrix representations of G over $\mathbf{K}$, and H be a matrix over $\mathbf{K}$. Assume that

$$T_1(g)H = HT_2(g), \qquad g \in G.$$

(1) If T_1 and T_2 are not equivalent, then $H = O$.

(2) If $\mathbf{K} = \mathbf{C}$ and $T_1 = T_2$, then $H = \alpha I$ for some $\alpha \in \mathbf{C}$.

We now define symmetrical homogeneous systems as follows.

Definition. A homogeneous system $(\bar{A}, \bar{B})$ is said to be *symmetric* for a specified group G if

$$T(g^{-1})\Delta T(g) = \Delta, \quad g \in G, \qquad (7)$$

where $T(g)$ is a matrix representation of G.
This equation (7) reflects the underlying geometric symmetry in the system structure since Δ contains all the structural information about system $(\bar{A}, \bar{B})$.

Lemma 1, together with the equations (6) and (7), shows that $Z^{-1}\Delta Z$ becomes block-diagonal with Z made by arranging $\{v^{\mu}_{ij}\}$: $Z^{-1}\Delta Z = \oplus_{\mu \in R(G)}(\Delta_\mu \otimes I_{n^\mu})$, where Δ_μ is of order a_μ. The basis $\{v^{\mu}_{ij}\}$ can be determined from $\{T^{\mu}(g) | \mu \in R(G)\}$ by means of the projection methods, see, e.g., 15).

Ring-type system is symmetric for the dihedral group D_m of order $2m$, which is defined as

$$D_m = \{1, r, \cdots, r^{(m-1)}; s, sr, \cdots, sr^{(m-1)}\},$$

where $r^m = s^2 = (sr)^2 = 1$. The irreducible representation decomposition is described as

$$T = \begin{cases} A_1 \oplus B_1 \oplus E_1 \oplus E_2 \oplus \cdots \oplus E_{\frac{m}{2}-1} & (m \text{ is even}), \\ A_1 \oplus E_1 \oplus E_2 \oplus \cdots \oplus E_{\frac{m-1}{2}} & (m \text{ is odd}), \end{cases}$$

where one-dimensional irreducible matrix representations are given as

$$T^{A_1}(g) = 1 \quad (g \in D_m)$$

and

$$\begin{cases} T^{B_1}(g) = 1 & (g \in \{1, sr^{2j}, r^{2j}, r^{m-2j}\}), \\ T^{B_1}(g) = -1 & (g \in \{sr^{2j+1}, r^{2j+1}, r^{m-2j-1}\}), \end{cases}$$

with $0 \le j \le \frac{m}{2} - 1$, and two-dimensional irreducible matrix representations are given as

$$T^{E_k}(r^l) = \begin{bmatrix} \cos kl\theta & -\sin kl\theta \\ \sin kl\theta & \cos kl\theta \end{bmatrix}$$

and

$$T^{E_k}(sr^l) = \begin{bmatrix} \cos kl\theta & -\sin kl\theta \\ -\sin kl\theta & -\cos kl\theta \end{bmatrix},$$

with $0 \le l \le (m-1)$. The first column of Z in (3) and (4) correpsonds to A_1 and the last of (4) to B_1. A pair of columns whose first entries are $\cos k\theta$ and $\sin k\theta$ corresponds to E_k. The matrix Z is thus obtained from the irreducible decomposition of the representation.

5. Conclusion

This paper has discussed the fault-tolerance of homogeneous systems with three basic and widely-used structures. As a measure of fault-tolerance, we adopt autonomous controllability, that is, controllability of the entire system in spite of the failures in some subsystems. The structure of homogeneous systems is characterized only by their interconnections among subsystems, and it contains no numerical information. We have shown the procedure to analyze the failure patterns that cause the systems to be generically uncontrollable merely by their structural information. And its result for three types of homogeneous systems has been shown. Moreover, symmetrical homogeneous systems have been defined by the use of group representation theory, which suggested that the analysis conducted in this paper can be further applied to various homogeneous systems with symmetrical structures.

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