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Graph-Dependent Sufficient Conditions for Synchronization of Network Coupled System with Time-delay

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This paper studies the synchronization of network coupled systems consisting of many identical dynamic subsystems as well as network coupling with interaction time-delay. Based on graph theory and Lyapunov stability theory, the paper gives two sufficient conditions for the total system synchronization with respect to the graph structures of network coupling interaction, one is delay-independent and the other is the delay-dependent. These two conditions are compared with Wu's research [6),7)], which was established without taking interaction time-delays into account. Simulations with two examples show the influences of the interaction time-delay as well as graph structures on the overall synchronization of subsystems.

Key Words: Network Coupled System, Time-delay, Graph, Synchronization

1. Introduction

As shown in many examples such as Benard Convection, Belousov-Zhabotinsky Reaction, group of automobiles and activity of biological cells, some kinds of spatiotemporal pattern are formed when many identical dynamical systems are coupled through the network coupling interaction. Even in case when dynamics of each subsystem itself are very simple, coupled systems generate the complex pattern. These interesting behaviors of network coupled systems, called as *complex systems* or *self-organization*, are studied in several regions and the application of these phenomena to the engineering is expected [1)]. An typical application of pattern formation on coupled systems for example is locomotion of legged robot using a central pattern generator [2),3)].

These behaviors of network coupled systems are strongly dependent on the coupling structure and the graph structure of network which constitutes interaction is an important factor because it determines the performance of coupled systems as well as interaction terms with each subsystems. The analysis of coupling structure to realize desired states of coupled systems have been performed by several researches. Yuasa and Ito formulated coupled systems whose subsystems dynamics are the gradient systems generated by potential functions defined on a graph as the coupling interaction and studied the control theory of coupled systems [4),5)]. Wu et al. gave a sufficient condition for synchronization of systems composed of identical dynamic subsystems and coupled through linear diffusive element. They reduce the condition to an eigenvalue problem of Laplacian matrix characterizing graph structure of network coupling interaction [6)]. Additionally, based on graph theory, they showed the effect of edges and vertices of graph on synchronization of coupled systems [7)].

As well as the coupling structure, time-delays within coupling interaction are also a typical factor characterizing coupled systems, since time-delays within coupling interaction often arises due to information communication and energy transportation [8)~10)]. As is well known, time-delays degrade the system performance and destabilize them, in the worst case when the controllers are designed without considering the time-delays [11)~13)].

This paper takes into account the time-delays within interaction and studies the synchronization of network coupled systems with respect to the graph structure of network and interaction terms with each system. For simplicity, in this paper only the case of a constant time-delay will be considered. We first formulate a mathematical model for the network coupled systems consisting of many identical dynamic subsystems. Following Wu, the network coupling interaction is represented in this model by a Laplacian matrix and a matrix representing the interaction strength in Kronecker product form. Based on stability analysis of delayed system using Lyapunov functional, we then derive delay-dependent as well as delay-independent sufficient conditions of the synchronization for coupled systems with respect to the coupling struc-

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ture. Two numerical examples are given to show the effectiveness of our studies. In first example, for a given graph of network coupling and interaction strength, the effect of time-delay's influence on the overall synchronization is compared with Wu's result, which does not take account for time-delay. The second example is given under different level of interaction strengths to show that, for two different types of coupling graph structures consisting of same numbers of subsystems, the graph structures influence the system synchronization in a nonlinear way.

In the following, problem formulation and preliminaries are given in section 2. The delay-independent synchronization condition is derived in section 3, and Section4 yields the delay-dependant one. We discuss our results in section 5 by using two numerical examples of networks consisting of many Chua's oscillators as its subsystem. Finally, Section 6 concludes the paper.

2. Problem Formulation and Preliminaries

Let us consider m identical dynamic subsystems that interact with one another via a network. Generally, the state equation of each subsystem is given by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i, t) \tag{1}$$

where $\boldsymbol{x}_i \in \mathbb{R}^n, i = 1, \cdots, m$ is the state vector. Following Wu's formulation [7] and taking into account about the time-delay within coupling interaction, the state equation of the whole system including network coupling interaction can be described as

$$\dot{\boldsymbol{x}} = \boldsymbol{I} \otimes \boldsymbol{f}(\boldsymbol{x}_i, t) + \boldsymbol{G}_0 \otimes \boldsymbol{D}_0 \boldsymbol{x} + \boldsymbol{G}_1 \otimes \boldsymbol{D}_1 \boldsymbol{x}(t-\tau) \tag{2}$$

$$\boldsymbol{G} \otimes \boldsymbol{D} = \begin{bmatrix} G_{11}\boldsymbol{D} & \cdots & G_{1m}\boldsymbol{D} \\ \vdots & \ddots & \vdots \\ G_{m1}\boldsymbol{D} & \cdots & G_{mm}\boldsymbol{D} \end{bmatrix}$$

where $\boldsymbol{x} = \left[\boldsymbol{x}_1^T \cdots \boldsymbol{x}_m^T\right]^T \in \mathbb{R}^{mn}$ is the state of the network coupled systems, $\boldsymbol{G}_0, \boldsymbol{G}_1 \in \mathbb{R}^{m\times m}$ are symmetric matrices solely determined as will be described below by the corresponding graph structure of the network, $\boldsymbol{D}_0, \boldsymbol{D}_1 \in \mathbb{R}^{n\times n}$ are real matrices denoting the coupling strength between subsystems, and $\tau > 0$ is a constant time-delay. $\otimes$ is Kronecker product (see appendix A for more information on its properties).

We shall study the synchronization of all subsystems with respect to the graph structure under the condition of interaction time-delay. We define the *synchronization* of the total system as follows.

Definition. A network coupled systems with state equation (2) synchronizes if the states of any two subsystems, $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, satisfy $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| \to 0$ as $t \to \infty$.

Associated with the graph structure of network interaction, the adjacency matrix $\boldsymbol{A}$ is defined in the following manner.

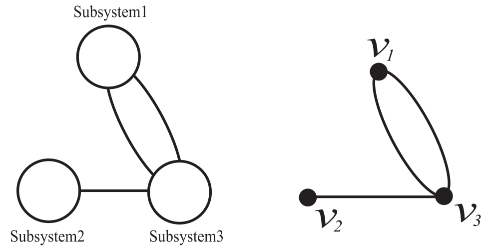


Fig. 1 Network system and Its Connectivity Graph

Definition. The adjacency matrix $\boldsymbol{A} = [A_{ij}]$ of a graph having p vertices is defined as follows: $A_{ij} = q$ if there are q edges connecting vertex i and vertex j in graph and $A_{ij} = 0$ otherwise. Here, we assume that the network does not include any self-loops.

Definition. For an $n \times m$ matrix $\boldsymbol{B}$, $\boldsymbol{\Phi}(\boldsymbol{B})$ is a diagonal $n \times n$ matrix, of which the diagonal elements are the row sum of $\boldsymbol{B}$.

Now, with above definitions, Laplacian matrix $\boldsymbol{G}$ is defined as follows.

Definition. Laplacian matrix $\boldsymbol{G}$ is constructed from matrix $\boldsymbol{A}$ and $\boldsymbol{\Phi}(\cdot)$ as: $\boldsymbol{G} = \boldsymbol{\Phi}(\boldsymbol{A}) - \boldsymbol{A}$.

An example of network coupled systems and its corresponding graph are shown in **Fig. 1**. Adjacency matrix $\boldsymbol{A}$ and the Laplacian matrix $\boldsymbol{G}$ of this system are as below:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \boldsymbol{G} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

Observe that $\boldsymbol{G}$ is symmetric, all of its off-diagonal elements are nonpositive and the sum of elements in each row is zero. These properties are easily verified by above definitions. With this in mind, we give the following definition.

Definition. By W, we denote the set of all irreducible matrices such that the off-diagonal elements are all nonpositive and the sum of elements in each row is zero.

If $\boldsymbol{G}$ in W, then the graph corresponding to this $\boldsymbol{G}$ is connected [14]. In this paper, it is assumed that $\boldsymbol{G}_0 + \boldsymbol{G}_1$ is *irreducible*, which means that the total system is connected through either time-delay or non-time-delay paths. However, $\boldsymbol{G}_0$ and $\boldsymbol{G}_1$ themselves may not be connected. In this case they can always be decomposed into matrices in W as:

$$G = C^T \begin{bmatrix} G'_1 & & & 0 \\ & \ddots & & \\ & & G'_h & \\ 0 & & & 0 \end{bmatrix} C, G'_1, \cdots, G'_h \in W \tag{3}$$

where C is a permutation matrix, and h is the number of connected graphs contained in the network. For matrix G in W, the following theorem gives important properties.

Theorem 2.1. [7] If $B \in \mathbb{R}^{m\times m}$ is a symmetric matrix in W, then B is positive semidefinite and has a zero eigenvalue with the corresponding eigenvector $[1 \ \cdots \ 1]^T$. Moreover, the zero eigenvalue has multiplicity 1.

3. Graph Structure and Delay-Independent Synchronization

In this and next sections, we study the synchronization of the system, focusing on the network graph structure together the time-delay. This section presents a synchronization condition which is independent of the time-delay. This will be accomplished by employing a Lyapunov functional [11)~13)].

3.1 Delay-independent synchronization condition

To begin with, we need the following definition on *V-uniformly decreasing function*.

Definition. [7] Given a square matrix V, a function $\phi(x,t) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is *V-uniformly decreasing* if there exits $c > 0$ such that for all x, y, t

$$(x-y)^T V(\phi(x,t) - \phi(y,t)) \le -c\,\|x-y\|^2 .$$

By assuming the *V-uniformly decreasing* property for subsystems (1), we give the following sufficient condition for synchronization of the network coupled systems (2):

Theorem 3.1. Let K be a matrix such that $f(x_i,t) + Kx_i$ is *V-uniformly decreasing* for some symmetric positive definite matrix V. If there exits a symmetric positive definite matrix Q and a symmetric matrix $U \in W$ such that

$$\begin{aligned}&(U\otimes V)(G_0\otimes D_0 - I\otimes K)\\ &+\frac{1}{2}(G_1\otimes D_1)^T Q (G_1\otimes D_1)\\ &+\frac{1}{2}(U\otimes V)^T Q^{-1}(U\otimes V) \le 0,\end{aligned} \tag{4}$$

then the system (2) synchronizes.

Proof. Define a Lyapunov functional g as follows:

$$\begin{aligned}g =& \frac{1}{2}x^T(U\otimes V)x + \frac{1}{2}\int_{-\tau}^{0} x^T(t+\theta)\\ &\times (G_1\otimes D_1)^T Q (G_1\otimes D_1) x(t+\theta)d\theta.\end{aligned} \tag{5}$$

Since the second term of (5) is positive, Lyapunov functional (5) is positive semidefinite and satisfies the following inequality:

$$g \ge \frac{1}{2}x^T(U\otimes V)x \ge 0.$$

By Theorem.2.1, U is positive semidefinite and has a zero eigenvalue of multiplicity 1 with eigenvector $[1 \ \cdots \ 1]^T$. Due to these properties of U and the positive definiteness of V, $g(x(\cdot))$ is zero if $x_i = x_j$ for all i, j, and is positive otherwise. The time derivative of g along trajectories of system (2) is:

$$\begin{aligned}\dot{g} =& x^T(U\otimes V)\{I\otimes f(x_i) + (I\otimes K)x\}\\ &+ x^T\Big\{(U\otimes V)(G_0\otimes D_0 - I\otimes K)\\ &+\frac{1}{2}(G_1\otimes D_1)^T Q(G_1\otimes D_1)\Big\}x\\ &+ x^T(U\otimes V)(G_1\otimes D_1)x(t-\tau)\\ &-\frac{1}{2}x^T(t-\tau)(G_1\otimes D_1)^T Q(G_1\otimes D_1)x(t-\tau).\end{aligned} \tag{6}$$

By the marix inequality (B. 1) in Appendix B, we get:

$$\begin{aligned}&2x^T(U\otimes V)(G_1\otimes D_1)x(t-\tau)\\ &\le x^T(U\otimes V)^T Q^{-1}(U\otimes V)x\\ &+ x^T(t-\tau)(G_1\otimes D_1)^T Q(G_1\otimes D_1)x(t-\tau).\end{aligned} \tag{7}$$

Moreover, from Wu's research [7], by the definition of *V-uniformly decreasing* and $U \in W$, we obtain:

$$\begin{aligned}&x^T(U\otimes V)\{I\otimes f(x_i) + (I\otimes K)x\}\\ &\le -c\alpha_{1i}\|x_1 - x_i\|^2 - \cdots - c\alpha_{jm}\|x_j - x_m\|^2\\ &= -cx^T(U\otimes I)x,\end{aligned} \tag{8}$$

where α_{jm} is a positive constant determined by matrix $U \in W$ [7]. Hence, it follows from (4),(6),(7) and (8) that

$$\dot{g} \le -cx^T(U\otimes I)x.$$

The time derivative of g is zero if $x_i = x_j$ for all i, j, and is negative otherwise. Thus, Lyapunov functional g along trajectories of system (5) approach to zero, and the states of any two subsystems, x_i and x_j, satisfy $\|x_i - x_j\| \to 0$ as $t \to \infty$. Therefore system synchronization is proved by Lyapunov's direct method. □

Note. For the network coupled systems without time-delay, Wu derived a sufficient condition for synchronization [7]

$$0 \ge (U\otimes V)(G_0\otimes D_0 + G_1\otimes D_1 - I\otimes K). \tag{9}$$

Compared this, since

$$
\begin{aligned}
0 \ge\, & (\boldsymbol{U} \otimes \boldsymbol{V})(\boldsymbol{G}_0 \otimes \boldsymbol{D}_0 - \boldsymbol{I} \otimes \boldsymbol{K}) \\
& + \frac{1}{2}(\boldsymbol{G}_1 \otimes \boldsymbol{D}_1)^T \boldsymbol{Q} (\boldsymbol{G}_1 \otimes \boldsymbol{D}_1) \\
& + \frac{1}{2}(\boldsymbol{U} \otimes \boldsymbol{V})^T \boldsymbol{Q}^{-1} (\boldsymbol{U} \otimes \boldsymbol{V}) \\
\ge\, & (\boldsymbol{U} \otimes \boldsymbol{V})(\boldsymbol{G}_0 \otimes \boldsymbol{D}_0 + \boldsymbol{G}_1 \otimes \boldsymbol{D}_1 - \boldsymbol{I} \otimes \boldsymbol{K}),
\end{aligned}
$$

condition (4) of Theorem.3.1 may look more conservative than the result of Wu. However, our condition covers the case constant interaction time-delay is included, which is not considered in the Wu's result [7].

3.2 Case study when $\boldsymbol{U} = \boldsymbol{G_0} + \boldsymbol{G_1} \in \boldsymbol{W}$

A key point in Theorem.3.1 is the selection of the matrix $\boldsymbol{U}$. An easy but most promised way is to set $\boldsymbol{U} = \boldsymbol{G}_0 + \boldsymbol{G}_1$. Note that by assumption, $\boldsymbol{G}_0 + \boldsymbol{G}_1$ belongs to W(See **Fig. 2**). Also, let us fix $\boldsymbol{Q}$ as $\boldsymbol{Q} = \boldsymbol{I}$. then we obtain from (4):

$$
\begin{aligned}
& \boldsymbol{G}_0^2 \otimes (\boldsymbol{V}\boldsymbol{D}_0 + \frac{1}{2}\boldsymbol{V}^2) + \boldsymbol{G}_1^2 \otimes (\frac{1}{2}\boldsymbol{D}_1^T\boldsymbol{D}_1 + \frac{1}{2}\boldsymbol{V}^2) \\
& + \boldsymbol{G}_1\boldsymbol{G}_0 \otimes (\boldsymbol{V}\boldsymbol{D}_0 + \frac{1}{2}\boldsymbol{V}^2) + \boldsymbol{G}_0\boldsymbol{G}_1 \otimes \frac{1}{2}\boldsymbol{V}^2 \\
& - (\boldsymbol{G}_0 + \boldsymbol{G}_1) \otimes \boldsymbol{V}\boldsymbol{K} \le 0.
\end{aligned}
$$

Using matrix inequality (B.1) in Appendix B, we have:

$$\boldsymbol{G}_0\boldsymbol{G}_1 + \boldsymbol{G}_1\boldsymbol{G}_0 \le \boldsymbol{G}_1\boldsymbol{G}_1 + \boldsymbol{G}_0\boldsymbol{G}_0.$$

Therefore, the condition for synchronization is converted to

$$\boldsymbol{M}_0 + \boldsymbol{M}_1 \le 0 \tag{10}$$

$$\boldsymbol{M}_0 = \boldsymbol{G}_0^2 \otimes (\frac{3}{2}\boldsymbol{V}\boldsymbol{D}_0 + \boldsymbol{V}^2) - \boldsymbol{G}_0 \otimes \boldsymbol{V}\boldsymbol{K} \tag{11}$$

$$\boldsymbol{M}_1 = \boldsymbol{G}_1^2 \otimes (\frac{1}{2}\boldsymbol{D}_1^T\boldsymbol{D}_1 + \boldsymbol{V}^2 + \frac{1}{2}\boldsymbol{V}\boldsymbol{D}_0) - \boldsymbol{G}_1 \otimes \boldsymbol{V}\boldsymbol{K}. \tag{12}$$

Thus the sufficient condition for this is obviously that both $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$ are negative semidefinite. By applying Theorem.Appendix A.1 in Appendix A to (11), (12), and noting that $\lambda(\boldsymbol{G}_0) \ge 0$, $\lambda(\boldsymbol{G}_1) \ge 0$, we then have the following lemma:

Lemma 3.1. Let $\boldsymbol{K}$ be a matrix such that $\boldsymbol{f}(\boldsymbol{x}_i, t)$ $+\boldsymbol{K}\boldsymbol{x}_i$ is $\boldsymbol{V}$-*uniformly decreasing* for some symmetric positive definite matrix $\boldsymbol{V}$. The system of (2) synchronizes if for all nonzero eigenvalues $\lambda_i(\boldsymbol{G}_0), \lambda_i(\boldsymbol{G}_1)$

$$
\begin{aligned}
& \lambda_i(\boldsymbol{G}_0)\left\{\frac{3}{2}(\boldsymbol{V}\boldsymbol{D}_0 + \boldsymbol{D}_0^T\boldsymbol{V}) + 2\boldsymbol{V}^2\right\} \\
& - (\boldsymbol{V}\boldsymbol{K} + \boldsymbol{V}\boldsymbol{K}^T) \le 0
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
& \lambda_i(\boldsymbol{G}_1)\left\{\boldsymbol{D}_1^T\boldsymbol{D}_1 + 2\boldsymbol{V}^2 + \frac{1}{2}(\boldsymbol{V}\boldsymbol{D}_0 + \boldsymbol{D}_0^T\boldsymbol{V})\right\} \\
& - (\boldsymbol{V}\boldsymbol{K} + \boldsymbol{V}\boldsymbol{K}^T) \le 0.
\end{aligned} \tag{14}
$$

Proof. By Theorem.Appendix A.1, the eigenvalues of $\boldsymbol{M}_0 + \boldsymbol{M}_0^T$ are the eigenvalues of $\lambda^2(\boldsymbol{G}_0)\left\{3/2(\boldsymbol{V}\boldsymbol{D}_0 + \boldsymbol{D}_0^T\boldsymbol{V}) +2\boldsymbol{V}^2\right\} - \lambda(\boldsymbol{G}_0)(\boldsymbol{V}\boldsymbol{K} + \boldsymbol{V}\boldsymbol{K}^T)$. Since $\lambda(\boldsymbol{G}_0) \ge 0$ and (13), $\boldsymbol{M}_0$ is negative semidefinite. Similarly $\boldsymbol{M}_1$ is negative semidefinite by $\lambda(\boldsymbol{G}_1) \ge 0$ and (14). Thus $\boldsymbol{M}_0 + \boldsymbol{M}_1$ is negative semidefinite. The lemma is proved. □

It is easy to see that if $\boldsymbol{V}\boldsymbol{D}_0 \le -(\boldsymbol{D}_1^T\boldsymbol{D}_1 + 2\boldsymbol{V}^2) \le 0$ and $\boldsymbol{V}\boldsymbol{K}$ is positive semidefinite, then system can synchronize by Lemma.3.1. However, in many case $\boldsymbol{V}\boldsymbol{K}$ is negative semidefinite, since there is trade-off between $\boldsymbol{K}$ and the $\boldsymbol{V}$-*uniformly decreasing* assumption of $\boldsymbol{f}(\boldsymbol{x}_i, t)$. Therefore it is preferable for system synchronization that $\boldsymbol{V}\boldsymbol{D}_0 \le -(\boldsymbol{D}_1^T\boldsymbol{D}_1 + 2\boldsymbol{V}^2 + \boldsymbol{X}) \le 0$, where $\boldsymbol{X}$ is positive semidefinite determined by $\lambda_i(\boldsymbol{G}_0)$, $\lambda_i(\boldsymbol{G}_1)$, and $\boldsymbol{K}$. Furhermore note that the upper bound of $\boldsymbol{D}_0$ includes not only $\lambda_i(\boldsymbol{G}_0)$, $\lambda_i(\boldsymbol{G}_1)$, and $\boldsymbol{K}$ but also $\boldsymbol{D}_1$. This implies that the increase of no-delayed coupling strength and the decrease of delayed one have the effect of preserving the synchronization.

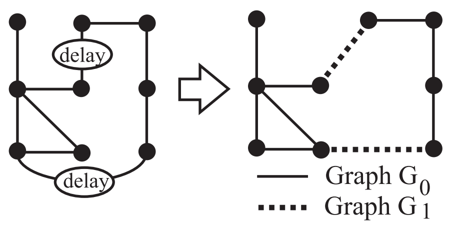


Fig. 2 Case: $\boldsymbol{G}_0 + \boldsymbol{G}_1 \in W$

4. Delay-Dependent Synchronization

The synchronization conditions derived in the last section are inherently conservative since they guarantees the synchronization regardless of the length of time-delay. In this section, we derive a delay-dependent synchronization condition for the network coupled systems (2). In Theorem.3.1, we assumed that the subsystem dynamics are $\boldsymbol{V}$-*uniformly decreasing*. In this section we have to further assume the Lipschitz continuity on (1).

Assumption 4.1. Function $\boldsymbol{f}(\boldsymbol{x}, t)$ is Lipschitz continuous in $\boldsymbol{x}$ with a Lipschitz constant γ, i.e.,

$$\|\boldsymbol{f}(\boldsymbol{x}, t) - \boldsymbol{f}(\boldsymbol{y}, t)\| \le \gamma \|\boldsymbol{x} - \boldsymbol{y}\| \quad \text{for all} \ \ \boldsymbol{x}, \boldsymbol{y}, t.$$

The following theorem gives the delay-dependent conditions for synchronization.

Theorem 4.1. Suppose that $\boldsymbol{f}(\boldsymbol{x}_i, t)$ is Lipschitz continuous in $\boldsymbol{x}$ with constant γ. Let $\boldsymbol{K}$ be a matrix such that $\boldsymbol{f}(\boldsymbol{x}_i, t)$ $+\boldsymbol{K}\boldsymbol{x}_i$ is $\boldsymbol{V}$-*uniformly decreasing* for some symmetric positive definite matrix $\boldsymbol{V}$. For given a scalar $\tau^* > 0$, if there exits a symmetric matrix $\boldsymbol{U} \in W$ and constants $r_1, r_2, r_3 > 0$ such that

$$(\boldsymbol{U} \otimes \boldsymbol{V})(\boldsymbol{G}_0 \otimes \boldsymbol{D}_0 + \boldsymbol{G}_1 \otimes \boldsymbol{D}_1 - \boldsymbol{I} \otimes \boldsymbol{K})$$

$$+\frac{1}{2}\tau^*\left\{r_1\gamma\lambda_{\max}(\boldsymbol{D}_1^T\boldsymbol{D}_1)\lambda_{\max}(\boldsymbol{G}_1)(\boldsymbol{G}_1\otimes\boldsymbol{I})\right.$$
$$+r_2(\boldsymbol{G}_0\otimes\boldsymbol{D}_0)^T(\boldsymbol{G}_0\otimes\boldsymbol{D}_0)$$
$$+r_3(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)^T(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)$$
$$+r_1^{-1}(\boldsymbol{U}\otimes\boldsymbol{V})(\boldsymbol{U}\otimes\boldsymbol{V})$$
$$+(r_2^{-1}+r_3^{-1})(\boldsymbol{U}\otimes\boldsymbol{V})(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)^T$$
$$\left.\times(\boldsymbol{U}\otimes\boldsymbol{V})\right\}\le 0, \tag{15}$$

then system (2) synchronizes for any constant delay-time $\tau\in[0,\tau^*]$.

Proof. Introduce the Lyapunov functional g as:

$$g=\frac{1}{2}\boldsymbol{x}^T(\boldsymbol{U}\otimes\boldsymbol{V})\boldsymbol{x}$$
$$+\frac{r_1}{2}\int_{-\tau}^{0}\int_{t+\theta}^{t}\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{F}(s)\|^2dsd\theta$$
$$+\frac{r_2}{2}\int_{-\tau}^{0}\int_{t+\theta}^{t}\|(\boldsymbol{G}_0\otimes\boldsymbol{D}_0)\boldsymbol{x}(s)\|^2dsd\theta$$
$$+\frac{r_3}{2}\int_{-\tau}^{0}\int_{t-\tau+\theta}^{t}\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{x}(s)\|^2dsd\theta, \tag{16}$$

where $\boldsymbol{F}(\cdot)=\boldsymbol{I}\otimes\boldsymbol{f}(\boldsymbol{x}_i(\cdot),\cdot)$. Following the same reasoning as in the proof of Theorem3.1, Lyapunov functional (16) is positive semidefinite and takes zero-value only if $\boldsymbol{x}_i=\boldsymbol{x}_j$ for all i,j. From (2), we have

$$\boldsymbol{x}(t-\tau)=\boldsymbol{x}(t)-\int_{-\tau}^{0}\dot{\boldsymbol{x}}(t+\theta)d\theta$$
$$=\boldsymbol{x}(t)-\int_{-\tau}^{0}\{\boldsymbol{F}(t+\theta)+(\boldsymbol{G}_0\otimes\boldsymbol{D}_0)\boldsymbol{x}(t+\theta)$$
$$+(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{x}(t-\tau+\theta)\}d\theta.$$

Substituting the above equation into (2), we get:

$$\dot{\boldsymbol{x}}=\boldsymbol{F}(t)+(\boldsymbol{G}_0\otimes\boldsymbol{D}_0+\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{x}$$
$$-(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\int_{-\tau}^{0}\boldsymbol{F}(t+\theta)d\theta$$
$$-(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\int_{-\tau}^{0}\boldsymbol{G}_0\otimes\boldsymbol{D}_0\boldsymbol{x}(t+\theta)d\theta$$
$$-(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\int_{-\tau}^{0}\boldsymbol{G}_1\otimes\boldsymbol{D}_1\boldsymbol{x}(t-\tau+\theta)d\theta. \tag{17}$$

Using this, the time derivative of g along the trajectories of (2) is computed as

$$\dot{g}=\boldsymbol{x}^T(\boldsymbol{U}\otimes\boldsymbol{V})\{\boldsymbol{F}(t)+(\boldsymbol{G}_0\otimes\boldsymbol{D}_0+\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{x}\}$$
$$-\boldsymbol{x}^T(\boldsymbol{U}\otimes\boldsymbol{V})(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\int_{-\tau}^{0}\{\boldsymbol{F}(t+\theta)$$
$$+(\boldsymbol{G}_0\otimes\boldsymbol{D}_0)\boldsymbol{x}(t+\theta)+(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{x}(t-\tau+\theta)\}d\theta$$
$$+\frac{1}{2}\tau r_1\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{F}(t)\|^2$$
$$-\frac{1}{2}\int_{-\tau}^{0}r_1\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{F}(t+\theta)\|^2d\theta$$
$$+\frac{1}{2}\tau r_2\|(\boldsymbol{G}_0\otimes\boldsymbol{D}_0)x\|^2$$
$$-\frac{1}{2}\int_{-\tau}^{0}r_2\|(\boldsymbol{G}_0\otimes\boldsymbol{D}_0)\boldsymbol{x}(t+\theta)\|^2d\theta$$
$$+\frac{1}{2}\tau r_3\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)x\|^2$$
$$-\frac{1}{2}\int_{-\tau}^{0}r_3\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{x}(t-\tau+\theta)\|^2d\theta.$$

To the second term, applying (B. 1) in appendix B, we have:

$$-2\boldsymbol{x}^T(\boldsymbol{U}\otimes\boldsymbol{V})(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\int_{-\tau}^{0}\boldsymbol{F}(t+\theta)d\theta$$
$$\le\int_{-\tau}^{0}\left\{r_1^{-1}\|(\boldsymbol{U}\otimes\boldsymbol{V})\boldsymbol{x}\|^2\right.$$
$$\left.+r_1\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)(\boldsymbol{F}(t+\theta))\|^2\right\}d\theta$$
$$=\tau r_1^{-1}\|(\boldsymbol{U}\otimes\boldsymbol{V})\boldsymbol{x}\|^2$$
$$+r_1\int_{-\tau}^{0}\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{F}(t+\theta)\|^2d\theta. \tag{18}$$

Similarly, applying (B. 1) to the other terms, we get

$$\dot{g}\le\boldsymbol{x}^T(\boldsymbol{U}\otimes\boldsymbol{V})\{\boldsymbol{F}(t)+(\boldsymbol{G}_0\otimes\boldsymbol{D}_0+\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{x}\}$$
$$+\frac{1}{2}\tau r_1\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{F}(t)\|^2+\frac{1}{2}\tau r_2\|(\boldsymbol{G}_0\otimes\boldsymbol{D}_0)x\|^2$$
$$+\frac{1}{2}\tau r_3\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)x\|^2+\frac{1}{2}\tau r_1^{-1}\|(\boldsymbol{U}\otimes\boldsymbol{V})x\|^2$$
$$+\frac{1}{2}\tau(r_2^{-1}+r_3^{-1})\|(\boldsymbol{U}\otimes\boldsymbol{V})(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)x\|^2. \tag{19}$$

By the spectral mapping theorem, all eigenvalues of $\boldsymbol{G}_1(\lambda_{\max}(\boldsymbol{G}_1)\boldsymbol{I}-\boldsymbol{G}_1)$ are nonnegative. Furthermore, since $\boldsymbol{f}(\boldsymbol{x}_i,t)$ is Lipschitz continuous in $\boldsymbol{x}$, the following relation for the norm of $\boldsymbol{F}(\cdot)$ holds:

$$\|(\boldsymbol{G}_1\otimes\boldsymbol{D}_1)\boldsymbol{F}(t)\|^2$$
$$\le\lambda_{\max}(\boldsymbol{D}_1^T\boldsymbol{D}_1)\lambda_{\max}(\boldsymbol{G}_1)\boldsymbol{F}^T(t)(\boldsymbol{G}_1\otimes\boldsymbol{I})\boldsymbol{F}(t)$$
$$\le\lambda_{\max}(\boldsymbol{D}_1^T\boldsymbol{D}_1)\lambda_{\max}(\boldsymbol{G}_1)\gamma\alpha_{1i}\|\boldsymbol{x}_1-\boldsymbol{x}_i\|+$$
$$\cdots+\lambda_{\max}(\boldsymbol{D}_1^T\boldsymbol{D}_1)\lambda_{\max}(\boldsymbol{G}_1)\gamma\alpha_{nj}\|\boldsymbol{x}_n-\boldsymbol{x}_j\|$$
$$=\lambda_{\max}(\boldsymbol{D}_1^T\boldsymbol{D}_1)\lambda_{\max}(\boldsymbol{G}_1)\gamma\boldsymbol{x}^T(\boldsymbol{G}_1\otimes\boldsymbol{I})\boldsymbol{x}. \tag{20}$$

Therefore using (8) in the previous section, (15), (19) and (20), then we have

$$\dot{g}\le -c\boldsymbol{x}^T(\boldsymbol{U}\otimes\boldsymbol{I})\boldsymbol{x}\le 0.$$

Hence, the time derivative of g is negative if the total system does not synchronize. The theorem is proved by Lyapunov's direct method. □

Note. Notice that when $\tau^*=0$, condition (15) reduces to the one derived by Wu [7]. Also note that τ^* appears linearly in (15) and the matrices that are included in the parentheses are non-negative definite. Therefore the larger the network time-delay τ^*, the more stringent the condition becomes for synchronization.

Note. To examine the influence of the network graph structure with time-delay, let us focus on the term $\boldsymbol{G}_1 \otimes \boldsymbol{D}_1$ in the synchronization condition (15) by assuming for simplicity that $\boldsymbol{D}_0 = 0$ and $\boldsymbol{D}_1 = \boldsymbol{D}x$, with $x \in \mathbb{R}^1$, a scalar that represents the interaction strength with time-delay. Denote the left side of (15) as $\boldsymbol{Y}$, then it is represented by the following parabola equation:

$$\boldsymbol{Y} = \boldsymbol{A}x^2 + \boldsymbol{B}x + \boldsymbol{C}$$

where

$$\begin{aligned}\boldsymbol{A} =& \frac{1}{2}\tau^* \left\{ r_1 \gamma \lambda_{\max}(\boldsymbol{D}^T\boldsymbol{D})\lambda_{\max}(\boldsymbol{G}_1)\left(\boldsymbol{G}_1 \otimes \boldsymbol{I}\right) \right.\\ &+ r_3 \left(\boldsymbol{G}_1 \otimes \boldsymbol{D}\right)^T \left(\boldsymbol{G}_1 \otimes \boldsymbol{D}\right)\\ &\left. + r_3^{-1}(\boldsymbol{U}\otimes\boldsymbol{V})(\boldsymbol{G}_1 \otimes \boldsymbol{D})(\boldsymbol{G}_1 \otimes \boldsymbol{D})^T(\boldsymbol{U}\otimes\boldsymbol{V}) \right\} \geq 0\\ \boldsymbol{B} =& (\boldsymbol{U}\otimes\boldsymbol{V})(\boldsymbol{G}_1 \otimes \boldsymbol{D})\\ \boldsymbol{C} =& \frac{1}{2}\tau^* r_1^{-1}\left(\boldsymbol{U}\otimes\boldsymbol{V}\right)\left(\boldsymbol{U}\otimes\boldsymbol{V}\right) - (\boldsymbol{U}\otimes\boldsymbol{V})(\boldsymbol{I}\otimes\boldsymbol{K}).\end{aligned}$$

Clearly, a coefficient matrix $\boldsymbol{A}$ of second order term in $\boldsymbol{Y}$ is positive semidefinite. Furthermore, if $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v} = 0$ for any vector $\boldsymbol{v} \neq 0$, then $\boldsymbol{v}^T\boldsymbol{B}\boldsymbol{v} = 0$. Therefore it is suggested that there will be an upper and lower bound x_0, x_1 for x within which $\boldsymbol{Y}$ becomes negative semidefinite.

Secondly, notice that if we introduce new edges or increase the number of edges connecting the subsystems, then the Laplacian matrices satisfy $\widehat{\boldsymbol{G}}_1 \geq \boldsymbol{G}_1$ where $\widehat{\boldsymbol{G}}_1$ denotes the Laplacian matrix corresponding to the graph with increased edge. This indicates that since $\boldsymbol{Y}$ depend on $\boldsymbol{G}_1$ binomially the increase of the time-delay edges in the graph influence the synchronization condition greatly. We will show some examples in the next section to verify these observations.

5. Numerical Simulations

Assume that subsystems are given by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i, t) = \begin{bmatrix} k\alpha(-x_{i1} + x_{i2} + h(x_{i1})) \\ k(x_{i1} - x_{i2} + x_{i3}) \\ k(-\beta x_{i2} - \mu x_{i3}) \end{bmatrix} \tag{21}$$

$$h(x_{i1}) = bx_{i1} + \frac{1}{2}(a-b)\left(|x_{i1}+1| - |x_{i1}-1|\right)$$

where parameters $k, \alpha, \beta, \mu > 0$ and $a > b > 0$. Equation (21) is known as a dimensionless Chua's oscillator, which exhibits chaotic phenomena as shown in **Fig. 3**, where the parameters are set as $k = 1, \alpha = 2, \beta = 2, \mu = 0.01, a = 1.14, b = 0.714$. To study the influences of the network's time-delay and the network graph structure to the overall system's synchronization, we perform following two numerical simulations.

5.1 Example 1

In this example, we fix the graph structure as shown in **Fig. 4** and will examine how does the time-delay influence the synchronization. Assuming that the interaction among subsystems arises only through the time-delayed edge, matrix $\boldsymbol{G}$ corresponding to the network graph is given as

$$\boldsymbol{G}_0 = \boldsymbol{0}, \boldsymbol{G}_1 = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Also choose the coupling strength matrix as

$$\boldsymbol{D}_0 = \boldsymbol{0}, \boldsymbol{D}_1 = \begin{bmatrix} -12 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -12 \end{bmatrix}.$$

By selecting parameters in Theorem.4.1 as

$$\boldsymbol{U} = \boldsymbol{G}_1,$$

$$\boldsymbol{V} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \boldsymbol{K} = \begin{bmatrix} -2.281 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0012 \end{bmatrix}$$

$$r_1 = 0.005, r_2 = 0, r_3 = 6.5,$$

it can be seen that for $\tau^* = 10.0$[ms] the synchronization conditions in Theorem.4.1 are all satisfied.

Figure 5 shows the time responses of the state difference $(\boldsymbol{x}_1 - \boldsymbol{x}_3)$ for the cases when there is no coupling as

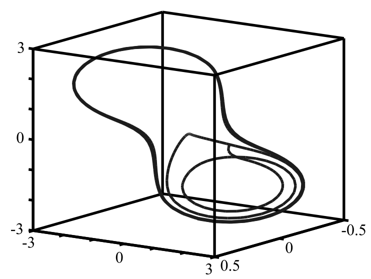


Fig. 3 Trajectory of a Chua's Oscillator

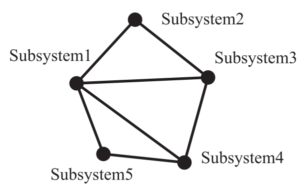
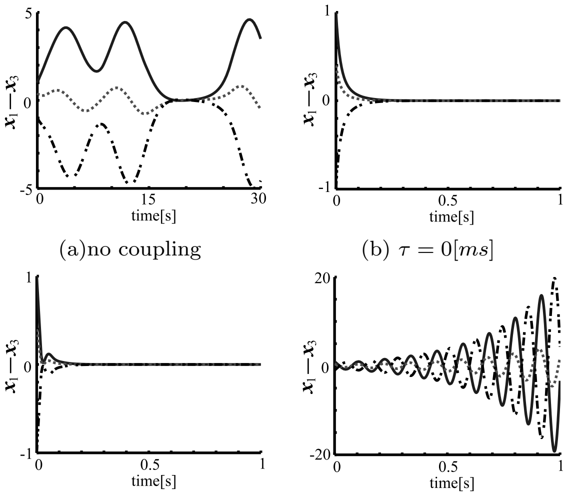


Fig. 4 Graph structure of simulation system



(a)no coupling (b) $\tau = 0[ms]$

(c)$\tau = 10[ms]$ (d) $\tau = 30[ms]$

Fig. 5 Time responses of the state differences $\boldsymbol{x}_1 - \boldsymbol{x}_3$ with $x_{11} - x_{31}$ (−), $x_{12} - x_{32}$ (·), $x_{13} - x_{33}$ (−·).

well as when the time-delays are $\tau = 0, 10$ and $30[ms]$, respectively. For $\tau = 0, 10[ms]$, the system is synchronizing as ensured by theorem.

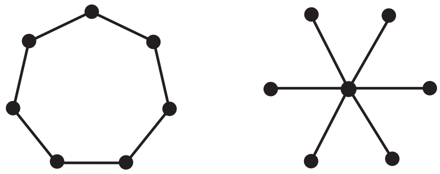
5.2 Example 2

In the second example, we compare two types of networks as shown in **Fig. 6**, one is the ring-graph and another is the star-graph. As in example 1, we assume that the interaction arises only through the time-delayed edge and set $\boldsymbol{D}_0 = \boldsymbol{0}$ and $\boldsymbol{D}_1 = -x\boldsymbol{I}$, with $x \in \mathbb{R}^1$, a scalar that represents the interaction strength.

The system synchronization results with respect to the changes of the time-delayed interaction strengths and the time-delays are summarized in **Table 1**, **2**, respectively. Upper and middle stands in Table 1, 2 show simulation results of each graph with respect to interaction strengths for $\tau = 0, 10$[ms]. Last stand shows the range of interaction strengths, which within the synchronization condition in the Theorem.4.1 holds for $\tau^* = 10$[ms]. ○ means that the system is synchronized and × says not. ⋆ means that synchronization of Theorem.4.1 condition holds.

From these results, the followings are observed. First, for the two types of the networks, when there is no time-delay ($\tau = 0$), both systems can realize synchronization if we simply increase the interaction strength x. Clearly, this property is suggested by our condition and Wu's result [7)].

Second, if there exists time-delay, with the ring-graph network, the system can only be synchronized for interaction strength within a range of $x \in [1\ 40]$ (in the case when $\tau = 10[ms]$), and for the star-graph, the range is $x \in [1\ 20]$, which is less than that of the ring-graph.



(a)Ring-graph structure (b) Star-graph structure

Fig. 6 Graph structures: Star-graph and Ring-graph

Table 1 The behavior of Ring-graph structure for each coupling strengths

x	0	1	2	3	4	5	10	20	30	40	50
$\tau = 0[ms]$	×	○	○	○	○	○	○	○	○	○	○
$\tau = 10[ms]$	×	○	○	○	○	○	○	○	○	○	×
$\tau^* = 10[ms]$					⋆	⋆	⋆	⋆			

Table 2 The behavior of Star-graph structure for each coupling strengths

x	0	1	2	3	4	5	10	20	30	40	50
$\tau = 0[ms]$	×	○	○	○	○	○	○	○	○	○	○
$\tau = 10[ms]$	×	○	○	○	○	○	○	○	×	×	×
$\tau^* = 10[ms]$				⋆	⋆	⋆	⋆				

These existences of upper and lower bound for interaction strength x in all simulation are consistent with our note in Section 4. However our conditions in this paper can not explain that there are differences of an interaction strength range for each graph structure, since the Lyapunov functional (16) includes not only Laplacian matrices $\boldsymbol{G}$ but also $\boldsymbol{U}$ which has to be chosen.

Third, the range of x with each graph for synchronization, which being calculated using Theorem.4.1, is smaller than that in this simulation. This implies that our synchronization conditions in Theorem.4.1 are still conservative. However, comparing with conditions in Theorem.3.1, which don't hold in this case ($\boldsymbol{D}_0 = \boldsymbol{0}$ and $\boldsymbol{D}_1 = -x\boldsymbol{I}$), conditions in Theorem.4.1 are less conservative.

6. Conclusions

In this paper, we have considered network coupled systems and derived two graph-dependent sufficient conditions, which ensure synchronization to occur among subsystems interconnected one another by constant time-delays. Of those two condition, one is delay-independent and the another is delay-dependent. The results are compared with Wu's research, which does not account for time-delay. According to our result, it is suggested that there will be a case where synchronization will occurs only when the interaction strength and the number of the graph-edges lie between some upper and lower bounds. This was verified with using an easy example.

A lot of studies remain to be studied. It is very important to extend the present result to systems containing multiple time-delays $\tau_1, \cdots, \tau_m$ as well as time-varying delays. Applications such as to multi-robots systems and bi-locomotion systems controlled by central pattern generators are also strongly anticipated.

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Appendix A. Kronecker Product and its Properties

The kronecker product has several properties as be listed below. Suppose that all operation of equations are valid.

- $(\boldsymbol{A} \otimes \boldsymbol{B})^T = \boldsymbol{A}^T \otimes \boldsymbol{B}^T$
- $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = (\boldsymbol{AC}) \otimes (\boldsymbol{BD})$
- Let $\lambda_i(\boldsymbol{A}), \lambda_j(\boldsymbol{B}), i = 1, \cdots, n, j = 1, \cdots, m$ denote the eigenvalues of $\boldsymbol{A} \in \mathbb{R}^{n\times n}, \boldsymbol{B} \in \mathbb{R}^{m\times m}$. Then the eigenvalues of $(\boldsymbol{A} \otimes \boldsymbol{B})$ are $\lambda_i(\boldsymbol{A})\lambda_j(\boldsymbol{B}), i = 1, \cdots, n, j = 1, \cdots, m$.

Theorem Appendix A.1. [7] Let p_1, p_2 be two polynominals. If $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are real symmetric matrices with eigenvectors $\boldsymbol{a}_i$, $\boldsymbol{b}_i$ and $\boldsymbol{c}_i$ and corresponding eigenvalues $\lambda_i(\boldsymbol{A})$, $\lambda_i(\boldsymbol{B})$ and $\lambda_i(\boldsymbol{C})$ respectively, then the symmetric matrix $p_1(\boldsymbol{A}) \otimes \boldsymbol{B} + p_2(\boldsymbol{A}) \otimes \boldsymbol{C}$ has eigenvectors $\boldsymbol{w}_{ij} = \boldsymbol{a}_i \otimes \boldsymbol{v}_{ij}$ with corresponding eigenvalues λ_{ij} where $\boldsymbol{v}_{ij}$ are the eigenvectors of $p_1(\lambda_i(\boldsymbol{A}))\boldsymbol{B} + p_2(\lambda_i(\boldsymbol{A}))\boldsymbol{C}$ with corresponding eigenvalues λ_{ij}.

Appendix B. Matrix Inequality

For any real matrices $\boldsymbol{A}, \boldsymbol{B}$, a symmetric positive definite matrix $\boldsymbol{C}$ with appropriate dimensions and a scalar $\varepsilon > 0$, the following equation hold:

$$\boldsymbol{A}^T\boldsymbol{B} + \boldsymbol{B}^T\boldsymbol{A} \le \varepsilon\boldsymbol{A}^T\boldsymbol{C}\boldsymbol{A} + \varepsilon^{-1}\boldsymbol{B}^T\boldsymbol{C}^{-1}\boldsymbol{B}. \quad \text{(B. 1)}$$

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