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An Observer Design and Separation Principle for the Motion of the n-dimensional Rigid Body

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In this paper an angular velocity observer for the motion of the n-dimensional rigid body is proposed. To describe the motion of rigid body, the Hamiltonian formulation is employed and no local coordinate is specified on Lie group $SO(n)$. Based on this approach, it is possible to focus on the intrinsic property of the system such as energy dissipation and to show closed-loop stability, a separation principle, which has been conjectured but not yet been shown.

Key Words: n-dimensional rigid body, geometric approach, Hamiltonian formulation, observer, separation principle

1. Introduction

The problem of controlling the motion of rigid bodies and mechanical linkages has been studied extensively in control, aerospace and robotics literature and has applications ranging from pointing and slewing maneuvers of spacecraft to object manipulation. A large amount of research has been carried out on the rigid body's attitude control problem [1), 15)]. It has been shown that passivity-based control, i.e. linear feedback of the position error and angular velocity with scalar gains, globally asymptotically stabilizes the origin of the closed-loop system [3), 13)]. However, angular velocity is not always measured in practice. For instance, small satellites are not equipped with gyros, angular velocity sensors, in recent trends because gyros are generally expensive and are often prone to degradation or failure. For such cases, an angular velocity observer of a rigid body from orientation and torque measurements was proposed [9)], but the closed-loop stability was not proven. Alternatively, the passivity-based, angular velocity-free set-point controller has been proposed [5), 14)].

It is well-known that the motion of a rigid body is represented by a set of two equations: (1) Euler's dynamic equation, which describes the time evolution of the angular velocity vector, and (2) the kinematic equation, which relates the time derivatives of the orientation angles and rotation group $SO(3)$ to the angular velocity vector. The important feature of the system is that its configuration space is $SO(3)$, which is not the Euclidean space but a manifold. Several parameterizations exist to represent the $SO(3)$, including three-parameter representations with singularity (e.g., Euler angles, Rodrigues parameters) and the four-parameter representation with an additional constraint without singularity (e.g., Euler parameters). Most research commonly involves the choice of a preliminary parameterization of coordinates for the configuration manifold $SO(3)$ [5), 9), 14), 15)]. By contrast, a coordinate-free approach is proposed for a trajectory tracking problem via differential geometric techniques [4)].

In this paper we deal with the free rotation of the n-dimensional rigid body about its center of mass on the Lie group $SO(n)$ in a coordinate-free framework by using the geometry of mechanical systems on manifolds. Avoiding the parameterization of the configuration space, it is possible to focus on the intrinsic property of the system. First, in section 2, we give Hamilton's canonical equations of an n-dimensional rigid body. Then, in section 3, we consider a set-point control problem of driving an attitude to a steady-state target attitude, and an angular velocity observer is obtained as a generalization of the Salcudean's observer [9)]. By taking errors of the plant and observer states as a ratio, the error dynamics also evolves on the same configuration space $SO(n)$. We remark that this is commonly observed in linear systems but not in nonlinear systems in general. Through this paper, it is seen that the approach taken enables us to see the geometric structure of the observer. Finally, in section 4, we solve the remaining problem, whether or not the observer-based controller still stabilizes the origin of the closed-loop system (separation principle). In section 5, we develop the above discussion into the global stabilization.

2. The n-Dimensional Rigid Body

In this section we introduce some notation and review

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some principal results on the kinematics and dynamics of the free rotation of an n-dimensional rigid body about a fixed point [6),7)]. The problem under consideration is the free rotation of an n-dimensional rigid body about its center of mass, which we assume to be the origin in $\mathbb{R}^n$. "Free" means that there are no external forces, and "rigid" means that the distance between any two points of the body is unchanged during the motion.

At first we consider the kinematics equation. Consider two coordinate systems: the body coordinate system and the spatial coordinate system. Throughout this paper, quantities expressed in the body coordinate system will be denoted by B, while quantities expressed in the spatial coordinate system will be denoted by S. Let $X_S(X_B, t) \in \mathbb{R}^n$ denote the position of the particle of the rigid body in spatial coordinate at time t which was at $X_B \in \mathbb{R}^n$ at time zero ($X_S(X_B, 0) = X_B$). Rigidity implies that $X_S(X_B, t) = q(t)X_B$, where $q(t) \in SO(n)$, the proper rotation group of $\mathbb{R}^n$, the $n \times n$ orthogonal matrices with determinant 1. The body and space coordinate velocity is

$$V_B(X_B, t) = -\frac{\partial X_B(X_S, t)}{\partial t} = q(t)^{-1}\dot{q}X_B(X_S, t)$$
$$V_S(X_S, t) = \frac{\partial X_S(X_B, t)}{\partial t} = q(t)V_B = \dot{q}(t)q(t)^{-1}X_S(X_B, t),$$

where $X_B(X_S, t) = q(t)^{-1}X_S$. We define $\omega_B(t) = q(t)^{-1}\dot{q}$, $\omega_S(t) = \dot{q}(t)q(t)^{-1} \in \mathfrak{so}(n)$, then ω_B, ω_S are left and right translations of $\dot{q} \in T_qSO(n)$ by Lie group $SO(n)$, and express $\dot{q}$ in body and space coordinates respectively (see Figure 1). ω_B, ω_S are called body and space coordinate angular velocity respectively. Thus kinematic equation is

$$\frac{dq(t)}{dt} = q(t)\omega_B(t) = \omega_S(t)q(t). \tag{1}$$

Next we consider the dynamics of the rigid body. Kinetic energy is conserved for the free rotation of an n-dimensional rigid body, then we derive the dynamic equation as Hamilton's equations in canonical coordinates. Let $\rho_0(X_B)$ be the mass density, the kinetic energy of the motion is obtained by summing up kinetic energy of each mass point over the body as follow

$$K(X_B) = \frac{1}{2}\int_{\mathcal{B}} \rho_0(X_B)\|V_B\|^2\, d^nX_B$$
$$= \frac{1}{2}\int_{\mathcal{B}} \rho_0(X_B)\|\omega_B(t)X_B\|^2\, d^nX_B.$$

For $\xi, \eta \in \mathfrak{so}(n)$, introducing the new inner product

$$\langle\langle \xi, \eta\rangle\rangle = \int \rho_0(X_B)\, X_B^T\xi^T\eta X_B\, d^nX_B,$$

the kinetic energy becomes

$$K(\omega_B) = \frac{1}{2}\langle\langle \omega_B, \omega_B\rangle\rangle.$$

Furthermore, introducing the following inner product on $\mathfrak{gl}(n, \mathbb{R})$, the vector space of all linear transformations of $\mathbb{R}^n$,

$$\langle A, B\rangle = \frac{1}{2}\mathrm{Trace}(A^TB), \quad A, B \in \mathfrak{gl}(n, \mathbb{R}),$$

and using

$$D_B = D_B^T = \int \rho_0(X_B)X_B{X_B}^T\, d^nX_B \geqslant 0,$$

the kinetic energy of the rigid body motion becomes

$$K(\omega_B) = \frac{1}{2}\langle\langle \omega_B, \omega_B\rangle\rangle = \frac{1}{2}\langle J_B(\omega_B), \omega_B\rangle \tag{2}$$
$$J_B(\cdot) : \xi \in \mathfrak{so}(n) \mapsto J_B(\xi) = D_B\xi - \xi^TD_B \in \mathfrak{so}(n),$$

where J_B is the moment of inertia tensor. Note that the inner product $\langle\cdot,\cdot\rangle$ is Ad_q invariant form ($\langle \mathrm{Ad}_q\xi, \mathrm{Ad}_q\eta\rangle = \langle \xi, \eta\rangle$) on $\mathfrak{so}(n)$, then $\langle\cdot,\cdot\rangle$ induces a left and right invariant Riemannian metric on $SO(n)$. Thus this metric defines a diffeomorphism $(\cdot)^\flat : TSO(n) \to T^*SO(n)$ by

$$(\cdot)^\flat : \nu_q \in T_qSO(n) \mapsto \nu_q^\flat = \langle \nu_q, \cdot\rangle \in T_q^*SO(n),$$

and its inverse is $(\cdot)^\sharp := (\cdot)^{\flat^{-1}} : T^*SO(n) \to TSO(n)$. From (1) and (2), Lagrangian $L : TSO(n) \to \mathbb{R}$ becomes

$$L(q, \dot{q}) = \frac{1}{2}\langle J_B(q^T\dot{q}), q^T\dot{q}\rangle.$$

Thus, angular momentum $p \in T_q^*SO(n)$, canonically conjugate to $q \in SO(n)$ is given by the Legendre transformation

$$p = \frac{\partial L}{\partial \dot{q}} = \langle qJ_B(q^T\dot{q}), \cdot\,\rangle = (qJ_B(q^T\dot{q}))^\flat,$$
$$\text{or} \quad p^\sharp = qJ_B(q^T\dot{q}).$$

Therefore, the Hamiltonian $H(q, p)$ and the dynamics equation $\dot{p}$ are

$$H(q, p) = \frac{1}{2}\langle q^Tp^\sharp, J_B^{-1}(q^Tp^\sharp)\rangle$$
$$\dot{p} = -\frac{\partial H}{\partial q} = \langle p^\sharp J_B^{-1}(q^Tp^\sharp), \cdot\,\rangle, \text{ or } \quad \dot{p}^\sharp = p^\sharp\, J_B^{-1}(q^Tp^\sharp),$$

respectively, where $J_B^{-1}(\cdot)$ is the inverse of $J_B(\cdot)$, and for $J_B(\eta) = \xi$, $\eta, \xi \in \mathfrak{so}(n)$ there exist a positive definite matrix E_B satisfying $J_B^{-1}(\xi) = E_B\xi - \xi^TE_B = \eta$. Note that when D_B is diagonalized, then E_B is always diagonalized for $n = 3$, but may not be diagonalized for general $n(> 3)$. And $J_B(\omega_B) = q^Tp^\sharp$, $J_S(\omega_S) = p^\sharp q^T \in \mathfrak{so}(n)$ denote the angular momentum corresponding to $p \in T_q^*SO(n)$ in body and spatial coordinate respectively, where $J_S(\xi_S) = \mathrm{Ad}_q(J_B(\xi)) = D_S\xi_S - {\xi_S}^TD_S \in \mathfrak{so}(n)$, $D_S = qD_Bq^T$, and $J_S^{-1}(\xi_S) = \mathrm{Ad}_q(J_B^{-1}(\xi)) = E_S\xi_S - {\xi_S}^TE_S$, $E_S = qE_Bq^T$, $\xi_S = \mathrm{Ad}_q\xi = q\xi q^T$. It should be noted that while J_B, D_B and E_B in the body frame are constant with time, J_S, D_S and E_S in the inertial frame are time varying.

We summarize:

Lemma 1 The Hamilton's canonical equations for the rotation of an n-dimensional rigid body with control inputs τ_H about its center of mass are

$$\Sigma_{HC}:\begin{cases} H(q,p) &= \frac{1}{2}\langle J_B^{-1}(q^T p^\sharp), q^T p^\sharp\rangle \\ \dfrac{dq}{dt} &= q\, J_B^{-1}(q^T p^\sharp) \\ \dfrac{dp^\sharp}{dt} &= p^\sharp\, J_B^{-1}(q^T p^\sharp) + \tau_H . \end{cases} \quad (3)$$

Using left and right translation (see Figure 1), we get from (3) the rigid body equations for body and spatial coordinates

$$\Sigma_{BC}:\begin{cases} H(q,\omega_B) &= \frac{1}{2}\langle \omega_B, J_B(\omega_B)\rangle \\ \dfrac{dq}{dt} &= q\,\omega_B \\ \dfrac{dJ_B(\omega_B)}{dt} &= [\,J_B(\omega_B),\,\omega_B\,] + q^T\tau_H \end{cases}$$

$$\Sigma_{SC}:\begin{cases} H(q,\omega_S) &= \frac{1}{2}\langle \omega_S, J_S(\omega_S)\rangle \\ \dfrac{dq}{dt} &= \omega_S\, q \\ \dfrac{dJ_S(\omega_S)}{dt} &= \tau_H q^T , \end{cases}$$

where $[\cdot,\cdot]$ denotes the Lie algebra bracket $[\xi,\eta]=\xi\eta-\eta\xi$, $\xi,\eta\in\mathfrak{so}(n)$, and the maps $\lambda:(q,\dot q)\mapsto(q,q^T\dot q)$ and $\rho:(q,\dot q)\mapsto(q,\dot q q^T)$ are left and right translation.

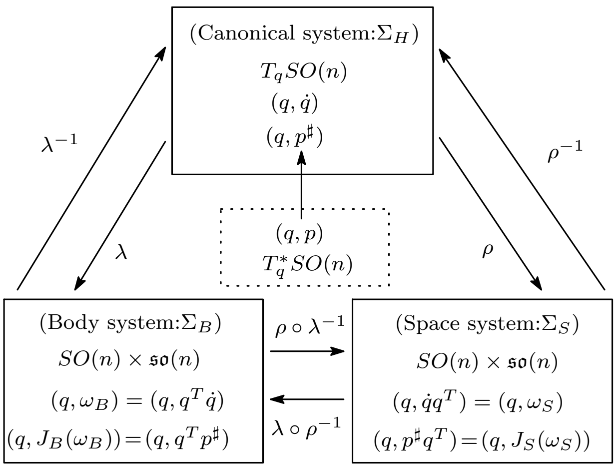

Fig. 1 Three coordinate systems.

3. Stabilizing Controller and Observer Design

3.1 Stabilizing Controller Design

We consider the set-point control problem of driving the initial rotating states $(q_0,p_0^\sharp)$ to a steady-state target attitude $(q_d,0)$. The following theorem is obtained from analogous to the well-known passivity based approach [13].

Theorem 2 For the rigid body control system Σ_{HC}, the control law

$$\tau_{HC}=-k_v\dot q-k_p(qq_d{}^Tq-q_d),$$

with positive constants $k_p,\,k_v>0$ asymptotically stabilizes the equilibrium $(q,p^\sharp)=(q_d,0)$, except for $q(\neq q_d)$ such that $qq_d{}^T=q_dq^T$. For the system Σ_{BC} and Σ_{SC}, the corresponding control law

$$\tau_{BC}=q^T\tau_{HC}=-k_v\omega_B-k_p(q_d{}^Tq-q^Tq_d)$$
$$\tau_{SC}=\tau_{HC}q^T=-k_v\omega_S-k_p(qq_d{}^T-q_dq^T)$$

asymptotically stabilize the equilibria $(q,J_B(\omega_B))=(q_d,0)$ and $(q,J_S(\omega_S))=(q_d,0)$ except for $q(\neq q_d)$ such that $qq_d{}^T=q_dq^T$, respectively.

Proof We give the proof of Σ_{HC}, the other cases are similar. Consider a Lyapunov function candidate

$$\begin{aligned} V_c &= H(q,p)+U_c(q) \\ &= \tfrac{1}{2}\langle J_B^{-1}(q^Tp^\sharp),q^Tp^\sharp\rangle + k_p\langle q_d-q\,,\,q_d-q\rangle, \end{aligned}$$

where the first term represents the kinetic energy and the second term represents the potential energy. We have

$$\frac{\partial U_c(q)}{\partial q}\cdot v=2k_p\langle q-q_d\,,\,v\rangle,\quad v\in T_qSO(n),$$

then the derivative of V_c along the trajectories of Σ_{HC} can be computed as

$$\begin{aligned} \dot V_1 &= \frac{\partial H}{\partial q}\cdot\dot q+\frac{\partial H}{\partial p^\sharp}\cdot\dot{p}^\sharp+\frac{\partial U_c}{\partial q}\cdot\dot q=\frac{\partial H}{\partial p^\sharp}\cdot\tau_{HC}+\frac{\partial U_c}{\partial q}\cdot\dot q \\ &= \langle\dot q\,,\,-k_v\dot q-k_p(qq_d{}^Tq-q_d)\rangle+2k_p\langle q-q_d\,,\,\dot q\rangle \\ &= -k_v\langle\dot q\,,\,\dot q\rangle-k_p\langle q^T\dot q\,,\,q_d{}^Tq+q^Tq_d-2I\rangle \\ &= -k_v\langle\dot q\,,\,\dot q\rangle=-k_v\|\dot q\|^2\leqslant 0, \end{aligned}$$

since $\langle\xi,A\rangle=\frac{1}{2}\mathrm{Trace}(\xi^TA)=0$ for all $A=A^T$, $\xi=-\xi^T\in\mathfrak{so}(n)$. Thus, LaSalle's Invariance Principle can be employed to complete the asymptotic stability. □

3.2 Observer Design

We deal with the problem of estimating the conjugate momentum p (or angular velocity ω) from the orientation $q\in SO(n)$ and torque measurements τ_H only. By taking errors of the plant and observer states as a ratio, the error dynamics also evolves on the same configuration space $SO(n)$. We make some remarks on this later. This nonlinear observer we proposed is a generalization of Salcudean's observer [9] to an n-dimensional rigid body in the Hamiltonian formulation.

Theorem 3 The n-dimensional rigid body observer for the Hamiltonian control system Σ_{HC} is

$$\Sigma_{HO}:\begin{cases}\dfrac{d\hat{q}}{dt} &= \hat{q}J_B^{-1}(q^T\hat{p}^\sharp\hat{q}^Tq)+u\\ \dfrac{d\hat{p}^\sharp}{dt} &= \hat{p}^\sharp J_B^{-1}(q^T\hat{p}^\sharp\hat{q}^Tq)+v_H\\ u &= -l_v(\hat{q}q^T\hat{q}-q)\\ v_H &= \tau_H q^T\hat{q}+l_v\hat{p}^\sharp(\hat{q}^Tq-q^T\hat{q})\\ &\quad +l_pJ_S^{-1}(q\hat{q}^T-\hat{q}q^T)\hat{q},\end{cases}$$

where l_p, $l_v > 0$ are positive constants, $(\hat{q},\hat{p}^\sharp)$ are estimated states of $(q,p^\sharp)$, and estimated states $(\hat{q},\hat{p}^\sharp)$ approach to $(q,p^\sharp)$ as $t\to\infty$ except for $\hat{q}(\neq q)$ such that $q\hat{q}^T=\hat{q}q^T$. By letting $\hat{\omega}_B=J_B^{-1}(q^T\hat{p}^\sharp\hat{q}^Tq)$, we have $\hat{q}^T\hat{p}^\sharp=\hat{q}^TqJ_B(\hat{\omega}_B)q^T\hat{q}$, then the corresponding observer for Σ_{BC} is

$$\Sigma_{BO}:\begin{cases}\dfrac{d\hat{q}}{dt} &= \hat{q}\hat{\omega}_B+u\\ \dfrac{d(\hat{q}^T\hat{p}^\sharp)}{dt} &= [\hat{q}^T\hat{p}^\sharp,\hat{\omega}_B]+v_B\\ v_B &= \hat{q}^Tq\tau_Bq^T\hat{q}+l_p\hat{q}^TqJ_B^{-1}(\hat{q}^Tq-q^T\hat{q})q^T\hat{q}\\ &\quad +l_v[\hat{q}^TqJ_B(\hat{\omega}_B)q^T\hat{q},\hat{q}^Tq-q^T\hat{q}].\end{cases}$$

By letting $\hat{\omega}_S=\mathrm{Ad}_q\hat{\omega}_B$, we have $\hat{\omega}_S=J_S^{-1}(\hat{p}^\sharp\hat{q}^T)$, $\hat{p}^\sharp\hat{q}^T=J_S(\hat{\omega}_S)$, then the corresponding observer for Σ_{BC} is

$$\Sigma_{SO}:\begin{cases}\dfrac{d\hat{q}}{dt} &= (\hat{q}q^T\hat{\omega}_Sq\hat{q}^T)\hat{q}+u\\ \dfrac{d(\hat{p}^\sharp\hat{q}^T)}{dt} &= v_S=\tau_S+l_pJ_S^{-1}(q\hat{q}^T-\hat{q}q^T).\end{cases}$$

Proof First, we consider the error dynamics. Since attitude and momentum of rigid body are elements of $SO(n)$ and $\mathfrak{so}(n)$ respectively, we choose errors between Σ_{HC} and Σ_{HO} to be also elements of $SO(n)$ and $\mathfrak{so}(n)$. To do this, we can choose error dynamics Σ_{SE} as follow

$$\Sigma_{SE}:\begin{cases}\dfrac{d(q\hat{q}^T)}{dt}=J_S^{-1}(p^\sharp q^T-\hat{p}^\sharp\hat{q}^T)q\hat{q}^T\\ \qquad\qquad -l_v(q\hat{q}^T-\hat{q}q^T)q\hat{q}^T\\ \dfrac{d(p^\sharp q^T-\hat{p}^\sharp\hat{q}^T)}{dt}=-l_pJ_S^{-1}(q\hat{q}^T-\hat{q}q^T).\end{cases}$$

Thus, the observer design reduces to stabilization of equilibrium $(q\hat{q}^T,p^\sharp q^T-\hat{p}^\sharp\hat{q}^T)=(I,0)$ of Σ_{SE}. Consider the Lyapunov function candidate V_o

$$V_o=\frac{1}{2}\langle p^\sharp q^T-\hat{p}^\sharp\hat{q}^T,p^\sharp q^T-\hat{p}^\sharp\hat{q}^T\rangle+l_p\langle I-q\hat{q}^T,I-q\hat{q}^T\rangle$$

Then, the time derivative of V_o along the trajectories of the error dynamics Σ_{SE} become

$$\begin{aligned}\dot{V}_o &= \langle p^\sharp q^T-\hat{p}^\sharp\hat{q}^T,\,\frac{d(p^\sharp q^T-\hat{p}^\sharp\hat{q}^T)}{dt}\rangle\\ &\quad -2l_p\langle I-q\hat{q}^T,\,\frac{d(q\hat{q}^T)}{dt}\rangle\\ &= -\langle p^\sharp q^T-\hat{p}^\sharp\hat{q}^T,l_pJ_S^{-1}(q\hat{q}^T-\hat{q}q^T)\rangle\\ &\quad -2l_p\langle\hat{q}q^T-I,J_S^{-1}(p^\sharp q^T-\hat{p}^\sharp\hat{q}^T)-l_v(q\hat{q}^T-\hat{q}q^T)\rangle\\ &= -l_p\langle J_S^{-1}(p^\sharp q^T-\hat{p}^\sharp\hat{q}^T),q\hat{q}^T+\hat{q}q^T-2I\rangle\\ &\quad -2l_vl_p\langle I-\hat{q}q^T,q\hat{q}^T-\hat{q}q^T\rangle\\ &= -l_vl_p\langle q\hat{q}^T-\hat{q}q^T,q\hat{q}^T-\hat{q}q^T\rangle\\ &= -l_vl_p\|q\hat{q}^T-\hat{q}q^T\|^2\leqslant 0.\end{aligned}$$

Using Barbalat's lemma, we see that $(q\hat{q}^T,p^\sharp q^T-\hat{p}^\sharp\hat{q}^T)\to(I,0)$ as $t\to\infty$ except for $\hat{q}(\neq q)$ such that $q\hat{q}^T=\hat{q}q^T$. □

Remark 1 If we write attitude error as $x=q\hat{q}^T\in SO(n)$ and angular velocity error as $\xi=\omega_S-\hat{\omega}_S\in\mathfrak{so}(n)$, the error dynamics with $l_p=l_v=0$ become

$$\Sigma_{SE}:\begin{cases}\dfrac{dx}{dt} &= \xi x\\ \dfrac{dJ_S(\xi)}{dt} &= 0,\end{cases}$$

which corresponds to the rigid body equation in space coordinates (Σ_{SC} with $\tau_H=0$). Thus stabilization of error dynamics ($(x,\xi)\to(I,0)$) is accomplished, first, adding the potential force $-l_pJ_S^{-1}(x-x^T)$, and next, the dissipation $-l_v(x-x^T)x$. We note that the mechanism of stabilization of the error dynamics is quite similar to that of Theorem 2 and that it is possible to see this picture because we avoid parameterizations of $SO(n)$ using geometric mechanics.

4. Observer-based Controller: Separation Principle

When the plant state is not available, the control law in Theorem 2 cannot be implemented. Therefore, one can consider using the observer-based control for the set-point control problem, but closed-loop stability has not been proved although angular velocity observer was proposed by Salcudean [9]. In this section, we show that it is possible in the stabilizing control law of Theorem 2 to replace $p^\sharp q^T$ by its estimate $\hat{p}^\sharp\hat{q}^T$ of Theorem 3 in the case where angular velocity of rigid body is not available. That is, it is shown that a separation principle-like property also holds for the nonlinear system considered in this paper, by avoiding parameterizations of $SO(n)$ and using the Hamiltonian formulation.

Theorem 4 Consider the closed-loop system Σ_{HC+HO} described by

$$\begin{cases} \dfrac{dq}{dt} &= q\, J_B^{-1}(q^T p^\sharp) \\ \dfrac{dp^\sharp}{dt} &= p^\sharp\, J_B^{-1}(q^T p^\sharp) + \tau_H \\ \dfrac{d\hat{q}}{dt} &= \hat{q} J_B^{-1}(q^T \hat{p}^\sharp \hat{q}^T q) + u \\ \dfrac{d\hat{p}^\sharp}{dt} &= \hat{p}^\sharp J_B^{-1}(q^T \hat{p}^\sharp \hat{q}^T q) + v_H \\ \tau_H &= -\, k_v q J_B^{-1}(q^T \hat{p}^\sharp \hat{q}^T q) - k_p(q{q_d}^T q - q_d) \end{cases},$$

where $k_p,\, k_v,\, l_p,\, l_v > 0$. Then the equilibrium $(q, p^\sharp, \hat{q}, \hat{p}^\sharp) = (q_d, 0, q, p^\sharp) = (q_d, 0, q_d, 0)$ of the system Σ_{HC+HO} is asymptotically stable.

Proof First, let us prove that the estimated states exponentially converge to the real states. We augment the Lyapunov function V_o used in Section 3.2 as:

$$W_{o\varepsilon} = V_o - \frac{1}{4}\varepsilon\big\langle p^\sharp q^T - \hat{p}^\sharp \hat{q}^T, J_S^{-1}(q\hat{q}^T - \hat{q}q^T)\big\rangle.$$

Rewriting $\mu = p^\sharp q^T - \hat{p}^\sharp \hat{q}^T$ and $\eta = q\hat{q}^T - \hat{q}q^T$, then the above becomes

$$W_{o\varepsilon} = \frac{1}{8}\mathrm{Tr}\left\{\begin{bmatrix} \mu \\ I - q\hat{q}^T \end{bmatrix}^T \begin{bmatrix} I & \varepsilon E_S \\ \varepsilon E_S & 2l_p \end{bmatrix} \begin{bmatrix} \mu \\ I - q\hat{q}^T \end{bmatrix}\right\} + \frac{1}{8}\mathrm{Tr}\left\{\begin{bmatrix} \mu \\ I - \hat{q}q^T \end{bmatrix}^T \begin{bmatrix} I & -\varepsilon E_S \\ -\varepsilon E_S & 2l_p \end{bmatrix} \begin{bmatrix} \mu \\ I - \hat{q}q^T \end{bmatrix}\right\}.$$

In addition, by calculating Schur complement of $\frac{3}{2}V_o - W_{o\varepsilon}$ and $W_{o\varepsilon} - \frac{1}{2}V_o$, we get

$$0 < \varepsilon < \sqrt{\frac{l_p}{2\lambda_{\max}({E_0}^2)}} \iff \frac{1}{2}V_o \leqslant W_{o\varepsilon} \leqslant \frac{3}{2}V_o. \qquad (4)$$

Then, the time derivative of $W_{o\varepsilon}$ along the trajectories of Σ_{SE} is

$$\begin{aligned} \dot{W}_{o\varepsilon} &= \dot{V}_o - \frac{1}{4}\varepsilon\Big\{\big\langle \dot{\mu}, J_S^{-1}(\eta)\big\rangle + \big\langle \mu, J_S^{-1}(\dot{\eta})\big\rangle + \big\langle \mu, \dot{J}_S^{-1}(\eta)\big\rangle\Big\} \\ &= -l_v l_p\langle \eta, \eta\rangle - \frac{1}{4}\varepsilon\Big\{-l_p\langle J_S^{-1}(\eta), J_S^{-1}(\eta)\rangle \\ &\quad + 2\langle J_S^{-1}(\mu), (J_S^{-1}(\mu) - l_v\eta)q\hat{q}^T\rangle + \langle \mu, \dot{J}_S^{-1}(\eta)\rangle\Big\}, \end{aligned}$$

where $\dot{J}_S^{-1}(\eta) = \dot{E}_S\eta + \eta\dot{E}_S$, $\dot{E}_S = \dot{q}E_B q^T + qE_B\dot{q}^T = J_S^{-1}(q^T p^\sharp)E_S - E_S J_S^{-1}(q^T p^\sharp) = \dot{E}_S^T$. Moreover, we consider the neighborhood of equilibrium such that

$$\|I - q\hat{q}^T\|^2 < 2. \qquad (5)$$

Then, by using inequalities in Proposition 7 of Appendix A, the above becomes

$$\begin{aligned} \dot{W}_{o\varepsilon} &\leqslant -l_p\big(l_v - \varepsilon\lambda_{\max}(E_B)^2\big)\|\eta\|^2 - \frac{\varepsilon}{2}\langle J_S^{-1}(\mu), J_S^{-1}(\mu)q\hat{q}^T\rangle \\ &\quad + \frac{\varepsilon}{2}l_v\|J_S^{-1}(\mu)\|\|\eta\| - \frac{\varepsilon}{4}\frac{\lambda_{\max}(\dot{E}_S)}{\lambda_{\min}(E_B)}\langle J_S^{-1}(\mu), \eta\rangle \\ &\leqslant -l_p\big(l_v - \varepsilon\lambda_{\max}(E_B)^2\big)\|\eta\|^2 - \frac{\varepsilon}{2}a\|J_S^{-1}(\mu)\|^2 \\ &\quad + \frac{\varepsilon}{2}l_v\|J_S^{-1}(\mu)\|\|\eta\| + \frac{\varepsilon}{4}\frac{\lambda_{\max}(\dot{E}_S)}{\lambda_{\min}(E_B)}\|J_S^{-1}(\mu)\|\|\eta\| \\ &\leqslant -\lambda_{\min}(P_o)\Big\{\|J_S^{-1}(\mu)\|^2 + \|\eta\|^2\Big\} \\ &\leqslant -\lambda_{\min}(P_o)\Big\{\|J_S^{-1}(\mu)\|^2 + \|I - q\hat{q}^T\|^2\Big\} \\ &\leqslant -\lambda_{\min}(P_o)\min\{8\lambda_{\min}(E_0)^2, 1/l_p\}V_o \\ &\leqslant -\frac{2}{3}\lambda_{\min}(P_o)\min\{8\lambda_{\min}(E_0)^2, 1/l_p\}W_{o\varepsilon} \leqslant 0, \end{aligned}$$

where

$$P_o = \begin{bmatrix} \dfrac{\varepsilon}{2}a & -\dfrac{\varepsilon}{4}\left(l_v + \dfrac{\lambda_{\max}(\dot{E}_S)}{2\lambda_{\min}(E_B)}\right) \\ -\dfrac{\varepsilon}{4}\left(l_v + \dfrac{\lambda_{\max}(\dot{E}_S)}{2\lambda_{\min}(E_B)}\right) & l_p\big(l_v - \varepsilon\lambda_{\max}(E_B)^2\big) \end{bmatrix}$$

satisfies $\lambda_{\min}(P_o) > 0$; *i.e.*,

$$l_p l_v > \varepsilon\left\{l_p\lambda_{\max}(E_B)^2 + \frac{1}{8a}\left(l_v + \frac{\lambda_{\max}(\dot{E}_S)}{2\lambda_{\min}(E_B)}\right)^2\right\}, \qquad (6)$$

and $0 < a < 1$ is a constant determined by using inequalities in Proposition 7 of Appendix A for $q\hat{q}^T$ satisfying (5). We summarize that if we choose ε small enough that conditions (4), (6) are satisfied, and if we also choose initial state such that (7) is satisfied, then condition (5) is satisfied and the observer error dynamics Σ_{SE} converges to equilibrium exponentially.

$$V_o(0) = \frac{1}{2}\|p^\sharp q^T - \hat{p}^\sharp \hat{q}^T\|^2 + l_p\|I - q\hat{q}^T\|^2\Big|_{t=0} < 2l_p. \qquad (7)$$

Finally, we show that the equilibrium of the closed-loop system Σ_{HC+HO} is asymptotically stable. Consider the following Lyapunov function candidate

$$V_{c+o} = \frac{2\lambda_{\min}(P_o)}{k_v}V_c + W_{o\varepsilon},$$

and evaluate its time derivative along Σ_{HC+HO}

$$\begin{aligned} \dot{V}_{c+o} &= \frac{2\lambda_{\min}(P_o)}{k_v}\dot{V}_c + \dot{W}_{o\varepsilon} \\ &\leqslant -\lambda_{\min}(P_o)\Big\{2\langle\, J_S^{-1}(p^\sharp q^T)\,,\, J_S^{-1}(\hat{p}^\sharp \hat{q}^T)\,\rangle \\ &\quad + \|J_S^{-1}(p^\sharp q^T - \hat{p}^\sharp \hat{q}^T)\|^2 + \|I - q\hat{q}^T\|^2\Big\} \\ &= -\lambda_{\min}(P_o)\Big\{\|J_S^{-1}(p^\sharp q^T)\|^2 + \|J_S^{-1}(\hat{p}^\sharp \hat{q}^T)\|^2 \\ &\quad + \|I - q\hat{q}^T\|^2\Big\} \leqslant 0. \end{aligned}$$

Then, we can choose $\alpha > 0$, $\varepsilon > 0$ such that (6), (7) are satisfied for $(q, p^\sharp, \hat{q}, \hat{p}^\sharp) \in {V_{c+o}}^{-1}([0, \alpha))$. Thus, by LaSalle's Invariance Principle, it follows that the equilibrium of the closed-loop system Σ_{HC+HO} is asymptotically stable except for $q(\neq q_d)$ such that $q{q_d}^T = q_d q^T$. □

5. Global Stability

The stability of Theorem 2, Theorem 3 and Theorem 4 is not a global one. This can be interpreted from the topological point of view. According to Milnor's theorem [10), 11)], on smooth globally asymptotically stable vector fields, the domain of attraction is a contractible set. The configuration space of rotation of an n-dimensional rigid body, that is, $SO(n)$ ($m = n(n-1)/2$ dimensional manifold) is not simply connected, thus, is not contractible. Therefore, it is not possible to find a continuous global stabilizing law.

To be onsistent with Milnor's theorem, and to achieve global stabilization, we introduce discontinuities in control law. We note that Euler-Rodrigues parameter is non-singular representation of $SO(3)$, then it is possible to achieve global stabilization for $n = 3$ as follow. Here we write $x^\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{so}(3)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Corollary 5 For $n = 3$, slight modifications of last term of τ_{HC}, v_H in Theorem 2, Theorem 3:

$$\begin{cases} -k_p q \dfrac{{q_d}^T q - q^T q_d}{\sqrt{1 + \mathrm{Trace}({q_d}^T q)}} & (\mathrm{Trace}({q_d}^T q) \neq -1) \\ -2k_p q n_1^\times & (\mathrm{Trace}({q_d}^T q) = -1) \end{cases}$$

$$\begin{cases} \dfrac{l_p J_S^{-1}(q\hat{q}^T - \hat{q}q^T)\hat{q}}{\sqrt{1 + \mathrm{Trace}(q\hat{q}^T)}} & (\mathrm{Trace}(q\hat{q}^T) \neq -1) \\ 2l_p J_S^{-1}(n_2^\times)\hat{q} & (\mathrm{Trace}(q\hat{q}^T) = -1) \end{cases}$$

achieve globally asymptotically stabilization in Σ_C, Σ_O, where n_1, n_2 are normalized eigenvectors of ${q_d}^T q$, $q\hat{q}^T$, respectively, whose eigenvalues are equal to 1. In this case, closed-loop system Σ_{C+O} using above observer-based controller is also globally asymptotically stabilized.

Proof The same procedure as proof of Theorem 2 , Theorem 3 and Theorem 4 apply to this proof as well. We can replace Lyapunov function V_c, V_o and $W_{o\varepsilon}$ by the following Lyapunov function candidate:

$$V_{c'} = \frac{1}{2}\langle J_B^{-1}(q^T p^\sharp), q^T p^\sharp\rangle + 2k_p\left(2 - \sqrt{4 - \|q - q_d\|^2}\right)$$

$$V_{o'} = \frac{1}{2}\langle p^\sharp q^T - \hat{p}^\sharp \hat{q}^T, p^\sharp q^T - \hat{p}^\sharp \hat{q}^T\rangle + 2l_p\left(2 - \sqrt{4 - \|I - q\hat{q}^T\|^2}\right)$$

$$W_{o'\varepsilon} = V_{o'} - \frac{1}{4}\varepsilon\langle p^\sharp q^T - \hat{p}^\sharp\hat{q}^T, J_S^{-1}(q\hat{q}^T - \hat{q}q^T),$$

respectively, then we get

$$\dot{V}_{c'} = -k_v\|\dot{q}\|^2$$

$$\dot{V}_{o'} = -l_p l_v \|q\hat{q}^T - \hat{q}q^T\|^2$$

$$\dot{W}_{o'\varepsilon} \leqslant -\frac{2}{3}\lambda_{\min}(P_o)\min\{8\lambda_{\min}(E_B)^2, 1/l_p\}W_{o'\varepsilon}$$

$$\left(0 < \varepsilon < \sqrt{\frac{l_p}{2\lambda_{\max}(E_B{}^2)}} \Longleftrightarrow \frac{1}{4}V_{o'} \leqslant W_{o'\varepsilon} \leqslant 3V_{o'}\right).$$

To complete the proof of the closed-loop stability, first, we can show that the equilibrium of closed-loop system is locally asymptotically stable in the same way as Theorem 4. Next, since error dynamics is globally asymptotically stable, for any initial state there exist a time T such that a similar condition to (7) is satisfied for all $t \geqslant T$, so the equilibrium is globally attractive. Thus, from the definition of stability, this shows that the equilibrium is globally asymptotically stable. □

Next, we consider the case of the general dimension. We do not have non-singular representations of $SO(n)$ except for $n = 3$ of Euler-Rodrigues parameter, then it may not be possible to generalize Corollary 5. Instead, for example, control law is forced to be updated as follow.

Corollary 6 For general n case, the control law in theorem 2 replaced by

$$\tau_H = \begin{cases} -k_v\dot{q} - k_p(q{q_d}^T q - q_d) & (\dot{q} \neq 0 \text{ or } q{q_d}^T \neq q_d q^T) \\ -k_p\|q - q_d\|q\xi & (\dot{q} = 0 \text{ and } q{q_d}^T = q_d q^T), \end{cases}$$

where $0 \neq \xi \in \mathfrak{so}(n)$, globally asymptotically stabilizes Σ_C. If observer estimates $(\hat{q}, \hat{p})$ in Theorem 3 are updated at specific instants of time t_1 such that $q_1(t_1)\hat{q}(t_1)^T = \hat{q}(t_1)q_1(t_1)^T$ to

$$(\hat{q}, \hat{p}) = (q(t_1), \hat{p}(t_1)\hat{q}(t_1)^T q(t_1)),$$

then Σ_O becomes globally stable observer, that is, the equilibrium of error dynamics is globally asymptotically stable. In this case, closed-loop system Σ_{C+O} using above observer-based controller is also globally asymptotically stabilized.

Proof One can check that the equilibrium of Σ_{HC}, Σ_{SE} is $(q, p^\sharp) = (q_d, 0)$, $(\hat{q}, \hat{p}^\sharp) = (q, p)$, respectively. Consider the same Lyapunov function candidate as Theorem 2, Theorem 3 and Theorem 4, then we get

$$\dot{V}_c = \begin{cases} -k_v\|\dot{q}\|^2 < 0 & (\dot{q} \neq 0) \\ 0 & (\dot{q} = 0) \end{cases}$$

$$\dot{V}_o = \begin{cases} -l_v l_p\|q\hat{q}^T - \hat{q}q^T\|^2 < 0 & (q\hat{q}^T \neq \hat{q}q^T) \\ 0 & (q\hat{q}^T = \hat{q}q^T), \end{cases}$$

thus, LaSalle's Invariance Principle [2)] can be employed to show that Σ_{HC}, Σ_{SE} is globally asymptotically stabilized. Using similar arguments as Corollary 5, we can show that closed-loop system is globally asymptotically stable. □

6. Conclusions

This paper was devoted to the attitude control problem and design of an angular velocity observer for the motion of the n-dimensional rigid body. Avoiding parameterizations of $SO(n)$ and using the Hamiltonian formulation, it was possible to reveal the geometric structure of the stabilizing controller and the angular velocity observer, and to demonstrate that the observer-based controller still stabilized the origin of the closed-loop system (separation principle).

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Appendix A. Supplement for Proof of Theorem 4

Proposition 7 The following inequalities are hold.

(1) $2\lambda_{\min}(E_B)\langle\xi,\xi\rangle \leqslant \langle J_S^{-1}(\xi),\xi\rangle \leqslant 2\lambda_{\max}(E_B)\langle\xi,\xi\rangle$

(2) $\|I-q\|^2 \leqslant 3 \Longrightarrow \|I-q\| \leqslant \|q-q^T\|$

(3) $\min_i(\cos\theta_i)\|\xi\|^2 \leqslant \langle\xi,\xi q\rangle$

where $\xi\in\mathfrak{so}(n)$, $q\in SO(n)$, and "normal form" [8)] of q is

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} \cos\theta_m & -\sin\theta_m \\ \sin\theta_m & \cos\theta_m \end{bmatrix} \end{bmatrix}$$

(n is odd; $n = 2m+1$).

Proof (1) E_B can be diagonalized, that is, $E = q_0^T E_B q_0 = \mathrm{diag}(e_1,\cdots,e_n)$ for some orthogonal matrix q_0. Let $\mathrm{Ad}_{(qq_0)^T}(\xi) = \xi_0$, and let ξ_{ij} be the (i,j)th matrix element of ξ_0, we get

$$\mathrm{Ad}_{(qq_0)^T}(J_S^{-1}(\xi)) = \mathrm{Ad}_{q_0^T} J_B^{-1}(\mathrm{Ad}_{q^T}\xi) = E\xi_0 + \xi_0 E.$$

Then, (i,j)th element of matrix $E\xi_0 + \xi_0 E$ becomes $(e_i+e_j)\xi_{ij}$. Therefore,

$$\begin{aligned}\langle J_S^{-1}(\xi),\xi\rangle &= \langle \mathrm{Ad}_{(qq_0)^T}(J_S^{-1}(\xi)), \mathrm{Ad}_{(qq_0)^T}(\xi)\rangle \\ &= \frac{1}{2}\sum_{i,j}(e_i+e_j)\xi_{ij}\xi_{ij} \leqslant 2\lambda_{\max}(E_B)\langle\xi,\xi\rangle.\end{aligned}$$

Similarly, $2\lambda_{\min}(E_B)\langle\xi,\xi\rangle \leqslant \langle J_S^{-1}(\xi),\xi\rangle$.

(2) For $n=3$, using normal form:

$$q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

we get

$$\begin{aligned}\|I-q\|^2 &= \mathrm{Tr}(I-q) = 2(1-\cos\theta) \\ \|q-q^T\|^2 &= \mathrm{Tr}(I-q^2) = (4-\|I-q\|^2)\|I-q\|^2.\end{aligned}$$

Moreover, using normal form of $SO(n)$, we have

$$\|I-q\|^2 \leqslant 3 \Longrightarrow \|I-q\| \leqslant \|q-q^T\|, \ \left({}^\forall q \in SO(n)\right).$$

(3) Since inner product $\langle\cdot,\cdot\rangle$ is Ad_q invariant on $\mathfrak{so}(n)$, using normal form of $SO(n)$, we get

$$\langle\xi,\xi q\rangle = \frac{1}{2}\langle\xi,\xi(q+q^T)\xi\rangle,$$

and this gives the statement of the proposition. □

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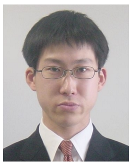

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