# A Numerical Algorithm of Discrete Fractional Calculus by using Inhomogeneous Sampling Data 

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#### Abstract

This paper presents an efficient numerical method to realize discrete models of fractional derivatives and integrals which imply derivatives and integrals of arbitrary real order. This approach is based on a class of Stieltjes integrals transferred from the Riemann-Liouville definition. It is to calculate on inhomogeneous sampling periods which are getting longer as the operation points go back toward the initial time. It leads to the effective quality which has low computational costs and enough accuracy. The calculation times and precision of the proposed procedure are compared with those of a conventional procedure for a practical numerical simulation and the effectiveness of this procedure is verified.


Key Words: fractional derivative, fractional integral, Stieltjes integral, discrete model, numerical algorithm

## 1. Introduction

Ordinarily, derivatives of functions are described in terms of integers, for example, the first derivative of position yielding speed, the second derivative yielding acceleration. The concept has been extended to non-integer derivatives, which are known as fractional derivatives (FD). Fractional integrals (FI) have also been defined, and these together make up fractional calculus (FC) ${ }^{1)}$. The concept of FC was known at the time of Leibniz during the 17 th century, but it has received little theoretical consideration since then, because its physical meaning was questionable and its mathematics was complicated. However, in recent years much research has been performed in physics, rheology, fractals, control systems and other fields where fractional differential equations made up of FD and FI have been reported to show several advantages ${ }^{2) \sim 4)}$.

In order to handle FC operators numerically, the systems must be transformed into discrete-time system models. It is easy to obtain very accurate and concise models of integer-order differential operators using Tustin transforms or other well-known trapezoidal integration methods. In contrast, FC operators are global with respect to time, so there is a strict requirement for the calculations to include the entire time history of the operators. If the size of the time step (i.e., sampling period) is small, however, the satisfaction of this requirement necessitates the use of large amounts of memory, which significantly increases the computational expense of the procedure.

[^0]It has been common to employ the G1-Algorithm ${ }^{1)}$ or any of the many other similar approaches based on the Grünwald-Letnikov differintegral, which is well suited for constructing algorithms suitable for numerical analysis ${ }^{5) \sim 11)}$. Recently, Podlubny ${ }^{12)}$ described both a geometrical interpretation and physical meaning of FD. Ma and Hori ${ }^{13)}$ employed these viewpoints in their proposal for a numerical solution method that has a clear physical analogue. However, these direct discrete procedures all suffer from high calculation costs imposed by the requirement of previous history. Practical implementations of these algorithms will need to get negotiate this problem by reducing the computational burden.

The Short Memory Principle ${ }^{4)}$, which fixes the range of calculations at some limited history and neglects whatever occurred earlier, is one possible approach for reducing the computations. However, the shorter the remaining calculation domain, the greater the error this principle introduces, so the researcher is forced to trade-off calculation time against precision.

This paper describes an inhomogeneous sampling algorithm (IS-Algorithm) as an essential solution for the issue of high computational costs ${ }^{14), 15)}$. It is based on the Riemann-Liouville integral, a class of Stieltjes integrals, which were used by Podlubny in the above publication to obtain some of his geometrical interpretations. In conventional methods, numerical integration is carried out in the so-called Riemann sum over the variable $t$ ("actual" time) at step $\Delta \tau$, i.e., in a discretized model with a constant sampling period. The present method, in contrast, employs the Stieltjes integral, performed over variable $T_{q}$ ("transformed" time) at a constant step $\Delta T$. The relation between "actual" time and "transformed" time is as
follows: When the "actual" time steps are constant, the "transformed" time is interpreted to pass at a gradually decreasing rate. Conversely, in this method, when the "transformed" time steps are set constant, the "actual" time steps do not remain constant in length, but become longer as one moves backward in time. As a result, fewer points in the history are sampled than in the conventional methods, greatly reducing the computational costs. Also, since the influence of the damping characteristics in the general case of physical dynamic phenomena diminishes as one goes backward in time, this method is less prone to lose computational precision, despite the large time steps for calculations involving the distant past.

This paper is organized as follows. Section 2 provides the definitions of the FC employed in this paper and describes the geometrical interpretations of those operations and also summarizes conventional discrete procedures. Section 3 shows the proposed discrete procedure for the new FC based on the geometrical interpretation introduced in Sec. 2. Section 4 compares the calculation steps of the proposed method with that of conventional procedures, and shows that the present procedure is effective at reducing the computational costs. An equation for estimating the error of the proposed procedure is also derived, and methods for estimating computational precision are discussed. Section 5 compares the calculation times and precision of the proposed procedure with those of a conventional procedure for a practical numerical simulation and verifies the effectiveness of this procedure. Finally, the results of this study are summarized.

## 2. Fractional Calculus

### 2.1 Definition of fractional calculus

In contrast to integer-order differentials $d^{n} / d t^{n}$, fractional-order differentials (FC) are defined as operators whose order has been extended to non-integer numbers. Several definitions have been proposed ${ }^{1), 4), 16)}$, but here the well-known Riemann-Liouville (R-L) and GrünwaldLetnikov (G-L) definitions are used.

The R-L definition ${ }^{4)}$ is used as the basis for the discrete procedure proposed in this paper. The definitions differ according to the sign in the FC, i.e., whether it is differentiation or integration. Let us first define the fractional integral (FI). The $q$-th ( $q>0, q \in \Re)$ FI of function $f(t)$ is given by

$$
\begin{equation*}
D^{-q} f(t) \triangleq \frac{1}{\Gamma(q)} \int_{t_{0}}^{t}(t-\tau)^{q-1} f(\tau) d \tau \tag{1}
\end{equation*}
$$

Here, $\Gamma(z)$ is the Gamma function:

$$
\begin{equation*}
\Gamma(z) \triangleq \int_{0}^{\infty} e^{-t} t^{z-1} d t, \quad(z>0) \tag{2}
\end{equation*}
$$

Eq.(1) above is used to define the fractional derivative (FD): The $q$-th ( $q \geq 0, q \in \Re)$ derivative of function $f(t)$ is given by

$$
\begin{align*}
D^{p} f(t) & \triangleq \frac{d^{m}}{d t^{m}}\left(D^{-(m-p)} f(t)\right) \\
& =\frac{1}{\Gamma(m-p)} \frac{d^{m}}{d t^{m}} \int_{t_{0}}^{t}(t-\tau)^{m-p-1} f(\tau) d \tau \tag{3}
\end{align*}
$$

where $m$ is a natural number satisfying $m-1 \leq q<m$.
The G-L definition ${ }^{1)}$ has been used as the basis for the G1-Algorithm, a typical example of conventional discrete procedures, and may be applied for assessing any arbitrary derivative. The $q$-th $(q \in \Re)$ FC of function $f(t)$ is defined as

$$
\begin{equation*}
D^{q} f(t) \triangleq \lim _{\substack{\Delta \tau \rightarrow 0 \\ \Delta \tau=\frac{t-t_{0}}{n}}} \Delta \tau^{-q} \sum_{j=0}^{n-1}(-1)^{j}\binom{q}{j} f(t-j \cdot \Delta \tau) \cdot(4) \tag{4}
\end{equation*}
$$

The above is a FD when $q>0$, and a FI when $q<0$, and the binomial coefficients are given in the general form:

$$
\begin{align*}
\binom{q}{0}=1, \quad\binom{q}{j} & =\frac{q(q-1) \cdots(q-j+1)}{j!}  \tag{5}\\
& =\frac{\Gamma(q-1)}{\Gamma(j+1) \Gamma(q-j+1)} .
\end{align*}
$$

### 2.2 Geometrical interpretation of the fractional integral

Let us describe geometrical interpretation of the FI presented by Podlubny ${ }^{12)}$, which is a foundation of the procedure suggested in this paper. The process here applies to the FI, but it may equally be applied to justify the FD.

The definition of the FI given by Eq.(1) can be rewritten in the form of a Stieltjes integral ${ }^{17)}$, which expresses a version of the ordinary Riemann form integration of $\tau$ as an integration of the variable $T_{q}(\tau)$, itself a function of $\tau$ :

$$
\begin{equation*}
D^{-q} f(t)=\int_{t_{0}}^{t} f(\tau) d T_{q}(\tau) \tag{6}
\end{equation*}
$$

The variable $T_{q}(\tau)$ is given by

$$
\begin{equation*}
T_{q}(\tau)=\frac{1}{\Gamma(q+1)}\left\{\left(t-t_{0}\right)^{q}-(t-\tau)^{q}\right\} . \tag{7}
\end{equation*}
$$

If we consider the variable of integration $\tau$ in Eq.(1) as the "actual" time, then the variable of integration $T_{q}(\tau)$ in Eq.(6) can be interpreted as the "transformed" time, while the degree of integration is here limited to $0<q \leq 1$ without loss of generality.
The geometrical interpretation of the Stieltjes integral in Eq.(6) is given in Figure 1. The function used


Fig. 1 The "fence" of $f(\tau)=\tau+0.5 \sin (2 \pi \tau)$ and its projection $D^{-1} f(t)$ and $D^{-0.5} f(t)$, for a constant "actual" time step $\Delta \tau$.
for the example in Fig. 1 is $f(\tau)=\tau+0.5 \sin (2 \pi \tau)$, $q=0.5,0 \leq \tau \leq 2$. A three-dimensional Cartesian space is drawn with axes $\left(\tau, T_{q}, f(\tau)\right)$ with the function in Eq.(7) graphed on the $\left(\tau, T_{q}\right)$ plane. Values of $f(\tau)$ are drawn vertically above the plane for this domain of $\tau$, to produce the curved "fence" in the figure. The area of this "fence" corresponds to the line integral along the integration path $T_{q}$ of function $f(\tau)$. The area projected from the "fence" onto the $(\tau, f)$ plane is the so-called "ordinary first integral" over $\tau$ :

$$
\begin{equation*}
D^{-1} f(t)=\int_{0}^{t} f(\tau) d \tau \tag{8}
\end{equation*}
$$

In contrast, the area of the projection on the $\left(T_{q}, f\right)$ plane corresponds to the value of the FI in Eq.(6) over integration variable $T_{q}$ (or in defining Eq.(1)).

As shown in the figure, if "actual" time $\tau$ is divided into constant time steps $\Delta \tau$, the corresponding steps $\Delta T$ of "transformed" time $T_{q}$ are not of constant length; the more distant in the past they are, the shorter they are (the finer the divisions), while the further they are in the future, the longer (coarser) they are.

### 2.3 Discretization procedure (G1-Algorithm) for conventional fractional calculus

One of the most simple and direct procedures for discretizing FC is given in Eq.(4) by simply omitting the limit notation. It is called the G1-Algorithm ${ }^{1)}$ :

$$
\left(D^{q} f(t)\right)_{\mathrm{G} 1}=\Delta \tau^{-q} \sum_{j=0}^{n-1}(-1)^{j}\binom{q}{j} f(t-j \cdot \Delta \tau),(9)
$$

where the symbol $(\cdot)_{\mathrm{G} 1}$ is used to designate the G1Algorithm. An equivalent of the above expression is
also obtained if a power series expansion (PSE) is carried out for the fractional differential operator $D^{q}\left(z^{-1}\right)=$ $\left(\left(1-z^{-1}\right) / \Delta \tau\right)^{q}$. An expression for the discretized model obtained by $z$-transform in the above equation is given by

$$
\begin{equation*}
\left(D^{q}\left(z^{-1}\right)\right)_{\mathrm{G} 1}=\Delta \tau^{-q} \sum_{j=0}^{n-1}(-1)^{j}\binom{q}{j} z^{-j} \tag{10}
\end{equation*}
$$

where $z^{-1} f(t)=f(t-\Delta \tau)$.
The sampling period $\Delta \tau$ in "actual" time $\tau$ is constant in conventional discretization procedures such as the G1Algorithm as shown in Fig.1. In other words, this means that the sampling period $\Delta T$ on the "transformed" time axis $T_{q}$ is not constant, but increases in length as time approaches the present, and, correspondingly, decreases in length as time moves further into the past. As a result, the greater the time span in the past for which this calculation is performed, the greater the computation is devoted to that past history, and so increasing the time span results in a extremely large increase in computational costs. The Short Memory Principle ${ }^{4)}$ and other approaches have been employed to overcome this problem. When the Short Memory Principle is employed, the function is not evaluated for the entire domain $\left[t_{0}, t\right]$, but rather, over some abbreviated "window" $[t-L, t]$ (see Fig.1) in order to prevent the demand for computational resources. However compressing the domain also increases computational error. This problem has hampered attempts to make the conventional procedures more efficient.

## 3. Inhomogeneous Sampling Algorithm

### 3.1 Procedure for discretizing the fractional integral by inhomogeneous sampling

This paper proposes an inhomogeneous sampling algorithm (IS-Algorithm) which promises to solve the problems faced by conventional procedures. The sampling period $\Delta T$ of "transformed" time $T_{q}(\tau)$ is set to a constant value, resulting in a discrete model with inhomogeneous "actual" time sampling period $\Delta \tau_{k}$.
Since the calculations of this procedure are performed in "actual" time, it is necessary to obtain the sampling points $\tau_{k}$, as shown below. Firstly, when the "transformed" time sampling period is set at $\Delta T$, we obtain the following relationship from Eq.(7):

$$
\begin{align*}
\Delta T & =T_{q}\left(\tau_{k-1}\right)-T_{q}\left(\tau_{k}\right) \\
& =\frac{1}{\Gamma(q+1)}\left\{\left(t-\tau_{k}\right)^{q}-\left(t-\tau_{k-1}\right)^{q}\right\} \tag{11}
\end{align*}
$$

If the initial time point $\tau_{0}$ is set equal to the present time,


Fig. 2 The function $f(\tau)$ is the same as Fig.1, but for the changing "actual" time steps $\Delta \tau$ and the constant "transformed" time step $\Delta T$.
$\tau_{0}=t$, then, from the above expression, the "actual" time sampling points $\tau_{k}(k=1,2, \cdots)$ are consecutively calculated thus:

$$
\begin{align*}
& \tau_{1}=t-\{\Delta T \cdot \Gamma(q+1)\}^{\frac{1}{q}} \\
& \tau_{2}=t-\{2 \Delta T \cdot \Gamma(q+1)\}^{\frac{1}{q}} \tag{12}
\end{align*}
$$

They are given in general by this expression:

$$
\begin{equation*}
\tau_{k}=t-K_{I} \cdot k^{\frac{1}{q}} \tag{13}
\end{equation*}
$$

in which

$$
\begin{equation*}
K_{I} \triangleq\{\Delta T \cdot \Gamma(q+1)\}^{\frac{1}{q}} \tag{14}
\end{equation*}
$$

Theorem 1. (Discretization of FI by IS-Algorithm)
By numerical integration methods such as the trapezoidal rule, the discrete model of IS-Algorithm is given by this expression:

$$
\begin{equation*}
\left(D^{-q} f(t)\right)_{\mathrm{IS}}=\sum_{k=1}^{m} \frac{f\left(\tau_{k-1}\right)+f\left(\tau_{k}\right)}{2} \Delta T \tag{15}
\end{equation*}
$$

where the symbol $(\cdot)_{\text {IS }}$ is used to designate the ISAlgorithm.

Proof. It is obtained by using the trapezoidal rule as an numerical integration method for Eq.(6).

The upper limit of summation $m$ can be calculated using Eq.(7):

$$
\begin{equation*}
m=\left[\frac{T_{q}(t)}{\Delta T}\right]=\left[\frac{\left(t-t_{0}\right)^{q}}{\Delta T \cdot \Gamma(q+1)}\right] \tag{16}
\end{equation*}
$$

where $[x]$ means truncation to the nearest integers. Similar to the conventional procedure, this procedure can use


Fig. 3 If the "transformed" time $T_{0.5}(\tau)$ ticks steady by the constant $\Delta T$, the "actual" time points $\Delta \tau_{k}$ become coarser in the past.
the Short Memory Principle ${ }^{4)}$, limiting the calculation window to some fixed time span $L$ (Fig.2). The upper limit of summation then takes the value

$$
\begin{equation*}
m=\left[\frac{L^{q}}{\Delta T \cdot \Gamma(q+1)}\right] \tag{17}
\end{equation*}
$$

If the discretization procedure in Eq.(15) is re-written as a model discretized using the $z$-transform, we obtain

$$
\begin{align*}
& \left(D^{-q}\left(z^{-1}\right)\right)_{\mathrm{IS}} \\
& \quad=\sum_{k=1}^{m} \frac{\Delta T}{2}\left\{z^{-\left[\frac{K_{I}}{\Delta \tau}(k-1)^{\frac{1}{q}}\right]}+z^{-\left[\frac{K_{I}}{\Delta \tau} \cdot k^{\frac{1}{q}}\right]}\right\}  \tag{18}\\
& \quad=\frac{\Delta T}{2}\left\{1+2 z^{-\left[\frac{K_{I}}{\Delta \tau}\right]}+2 z^{-\left[\frac{K_{I}}{\Delta \tau} \cdot 2^{\frac{1}{q}}\right]}+\cdots\right\}
\end{align*}
$$

Let us now consider specific design parameters for this procedure: order of integral $q$, sampling period $\Delta T$ in "transformed" time, and calculation window length $L$ if necessary. When the sampling period in "actual" time $\Delta \tau=0.1$, for example, then, setting the parameters $q=0.5, \Delta T=0.1$ determines $K_{I} \cong 7.9 \times 10^{-3}$. The full model expression would then be written as:

$$
\begin{align*}
\left(D^{-0.5}\right. & \left.\left(z^{-1}\right)\right)_{\mathrm{IS}}=\frac{\Delta T}{2}\left(1+2 z^{-[0.08]}+2 z^{-[0.31]}\right. \\
& +2 z^{-[0.71]}+2 z^{-[1.26]}+2 z^{-[1.96]}  \tag{19}\\
& \left.+2 z^{-[2.83]}+2 z^{-[3.85]}+2 z^{-[5.03]}+\cdots\right) \\
& =0.05\left(7+4 z^{-1}+2 z^{-2}+2 z^{-3}+2 z^{-5}+\cdots\right)
\end{align*}
$$

Figure 3 represents the relationship between the two time


Fig. 4 The slopes of the transformed time $T_{q}(\tau)$ vary corresponding to the values of $q$ varying from 0 to 1 .
axes $\tau$ and $T_{q}$ for $q=0.5$ of Eq.(7) which result the values of $\tau_{k}$ from a constant $\Delta T$. Thus, the periods between consecutive sampling points $\tau_{k}$ become shorter as time approaches the present, while the points are getting coarser in the past. This reduces the rate of growth in number of sampling points even though the overall time span is extended, preventing the extreme increase in computational costs.

Figure 4 also represents the slopes of the transformed time $T_{q}(\tau)$ corresponding to the values of $q$ varying from 0 to 1 . It describes that the smaller the order, the higher the performance of reducing the computational costs. It is known that several real systems such as the viscoelastic materials have the value of their order of derivative around $0.5^{18)}$.

### 3.2 Discretization procedure for FD by inhomogeneous sampling

The R-L definition of the FD (Eq.(3)) involves taking an integer-order derivative of the FI, so the proposed FD discretization procedure consists of the simple backward difference of the FI described in the previous chapter. We shall therefore skip over the portion of the procedure which is identical to that followed in the preceding chapter and proceed to the main results. If we limit the order of the differential to $0 \leq q<1$, then, without loss of generality, the FD in Eq.(3) becomes

$$
\begin{equation*}
D^{q} f(t)=\frac{d}{d t} \int_{t_{0}}^{t} f(\tau) d T_{1-q}(\tau) \tag{20}
\end{equation*}
$$

Theorem 2. (Discretization of FD by IS-Algorithm) The trapezoidal calculation for the FD corresponding to Eq.(15), becomes

$$
\begin{equation*}
\left(D^{q} f(t)\right)_{\mathrm{IS}}=\sum_{k=1}^{l} \frac{\dot{f}\left(\nu_{k-1}\right)+\dot{f}\left(\nu_{k}\right)}{2} \Delta T \tag{21}
\end{equation*}
$$

where

$$
\begin{align*}
& \dot{f}\left(\nu_{k}\right) \triangleq \frac{f\left(\nu_{k}\right)-f\left(\nu_{k}-\Delta \tau\right)}{\Delta \tau}  \tag{22}\\
& \nu_{k}=t-K_{D} \cdot k^{\frac{1}{1-q}}  \tag{23}\\
& K_{D}=\{\Delta T \cdot \Gamma(2-q)\}^{\frac{1}{1-q}} \tag{24}
\end{align*}
$$

Proof. Eq.(21) is obtained by using the trapezoidal rule as an numerical integration method for Eq.(20).

The upper limit of summation $l$ is

$$
l=\left[\left(t-t_{0}\right)^{1-q} /(\Delta T \cdot \Gamma(2-q))\right]
$$

when the calculation is carried out over the entire domain, and

$$
l=\left[L^{1-q} /(\Delta T \cdot \Gamma(2-q))\right]
$$

when it is carried out only over window $L$.
The above is then re-written as a $z$-transform model:

$$
\begin{align*}
& \left(D^{p}\left(z^{-1}\right)\right)_{\text {IS }} \\
& =\sum_{k=1}^{l} \frac{\Delta T}{2} \frac{1-z^{-1}}{\Delta \tau}\left\{z^{-\left[\frac{K_{D}}{\Delta \tau}(k-1)^{\frac{1}{1-p}}\right]}+z^{-\left[\frac{K_{D}}{\Delta \tau} \cdot k^{\frac{1}{1-p}}\right]}\right\} \\
& =\frac{\Delta T}{2 \Delta \tau}\left\{1-z^{-1}+2\left(1-z^{-1}\right) z^{-\left[\frac{K_{D}}{\Delta \tau}\right]}\right.  \tag{25}\\
& \left.\quad+2\left(1-z^{-1}\right) z^{-\left[\frac{K_{D}}{\Delta \tau} \cdot 2^{\frac{1}{1-p}}\right]}+2\left(1-z^{-1}\right) z^{-\left[\frac{\left.K_{D} \cdot 3^{\frac{1}{1-p}}\right]}{\Delta \tau}\right.}+\cdots\right\}
\end{align*}
$$

If the sampling period $\Delta \tau$ is set at 0.1 , for example, and the design parameters are set to $q=0.5$ and $\Delta T=0.1$, the model is then expressed thus:

$$
\begin{align*}
& \left(D^{0.5}\left(z^{-1}\right)\right)_{\text {IS }}=\left(0.5-0.5 z^{-1}\right) \\
& \quad+\left(z^{-[0.08]}-z^{-1-[0.08]}\right)+\left(z^{-[0.31]}-z^{-1-[0.31]}\right) \\
& \quad+\left(z^{-[0.71]}-z^{-1-[0.71]}\right)+\left(z^{-[1.26]}-z^{-1-[1.26]}\right) \\
& \quad+\left(z^{-[1.96]}-z^{-1-[1.96]}\right)+\left(z^{-[2.83]}-z^{-1-[2.83]}\right) \\
& \quad+\left(z^{-[3.85]}-z^{-1-[3.85]}\right)+\left(z^{-[5.03]}-z^{-1-[5.03]}\right) \\
& \quad+\left(z^{-[6.36]}-z^{-1-[6.36]}\right)+\left(z^{-[7.85]}-z^{-1-[7.85]}\right) \\
& =3.5-1.5 z^{-1}-z^{-2}-z^{-4}+z^{-5}-z^{-8}+\cdots . \tag{26}
\end{align*}
$$

## 4. Assessment of the Proposed Procedure

### 4.1 Comparison of calculation steps

Let us compare the number of calculations needed by the conventional procedures with those needed by the procedure proposed here. For simplicity, the following discussion counts a single time step for any one of the numerical


Fig. 5 Comparison between the proposed method's calculation steps and the conventional one.
integrations as a single iteration. The single iterations actually would consume different amounts of time, however, since they consist of various operations - calculation of binomial coefficients for the conventional procedure, and algebraic computations for the trapezoidal or other methods of the proposed technique.

For the case of conventional computations (G1), if the calculation time is $t-t_{0}$ and the sampling period is $\Delta \tau$, then the total number of samples is $n=\left(t-t_{0}\right) / \Delta \tau$. The number of calculations necessary for the integration of the $k$-th point is $k$, so the total number of calculations $\left(N_{c}\right)$ of G1 is

$$
\begin{align*}
\left(N_{c}\right)_{\mathrm{G} 1} & =\sum_{k=1}^{n} k=\frac{t-t_{0}}{2 \Delta \tau}\left(\frac{t-t_{0}}{\Delta \tau}+1\right) \\
& =\frac{\left(t-t_{0}\right)^{2}}{2 \Delta \tau^{2}}\left(1+\frac{\Delta \tau}{t-t_{0}}\right) \tag{27}
\end{align*}
$$

In the proposed procedure, however, the number of calculations required for integration at the $k$-th point is $T(k \cdot \Delta \tau) / \Delta T$. Substituting this relationship in Eq.(7), we find the total number of calculations $\left(N_{c}\right)$ of IS is

$$
\begin{align*}
\left(N_{c}\right)_{\mathrm{IS}} & =\sum_{k=1}^{n} \frac{T(k \cdot \Delta \tau)}{\Delta T}=\sum_{k=1}^{\frac{t-t_{0}}{\Delta \tau}}\left[\frac{(k \cdot \Delta \tau)^{q}}{\Gamma(q+1) \cdot \Delta T}\right]  \tag{28}\\
& \cong \frac{\left(t-t_{0}\right)^{q+1}}{\Gamma(q+1) \cdot \Delta T \cdot \Delta \tau}\left\{\frac{1}{q+1}+\frac{\Delta \tau}{2\left(t-t_{0}\right)}\right\}
\end{align*}
$$

The final approximation of Eq.(28) is obtained by substituting in the approximation for the summation

$$
\begin{equation*}
\sum_{k=1}^{n} k^{q} \cong n^{q+1}\left(\frac{1}{q+1}+\frac{1}{2 n}\right) \tag{29}
\end{equation*}
$$

Figure 5 shows the numbers of calculations specifically required by each method when, for example, the degree of
integration is set at $q=0.5, \Delta \tau=\Delta T=0.01$, and the calculation window $t-t_{0}=100[\mathrm{~s}]$. Thus, $N_{c}$ is observed to increase with the square of the number of sampling points in the conventional method, i.e. $N_{c} \sim O\left(n^{2}\right)$, while the proposed method suppresses the rate of increase; it is a more efficient calculation method.

## 4. 2 Assessing error in the proposed method

The accuracy of the calculations performed by the proposed method was also investigated. The operation examined here is FI, but the results are equally applicable to FD. Since the proposed method is numerical, uses trapezoids and is based in the Stieltjes integral in Eq.(6), the Euler-Maclaurin formula for the error of the trapezoidal rule ${ }^{19)}$ can be applied.

Lemma 1. (Assessment of the error of the trapezoidal rule) Let $f(x) \in C^{2}[a, b], n=(b-a) / h$. Then, the error given by the trapezoidal rule will be given by

$$
\begin{align*}
\varepsilon & =\left|\int_{a}^{b} f(x) d x-\frac{h}{2} \sum_{k=1}^{n}\{f(a+(k-1) h)+f(a+k h)\}\right| \\
& \leq \frac{b-a}{12} h^{2} \max _{a \leq \theta \leq b}\left|f^{(2)}(\theta)\right| \tag{30}
\end{align*}
$$

Proof. See 19).
The total error of IS-Algorithm is given by a theorem below.

Theorem 3. (Assessment of the error of the ISAlgorithm)

$$
\begin{align*}
\varepsilon= & \left|D^{-q} f(t)-\left(D^{-q} f(t)\right)_{\mathrm{IS}}\right| \\
\leq & \frac{\left(t-t_{0}\right)^{q+1} \cdot \Delta T^{2}}{12 \cdot \Gamma(q+1) \cdot \Delta \tau}  \tag{31}\\
& \left\{\frac{1}{q+1}+\frac{\Delta \tau}{2\left(t-t_{0}\right)}\right\} \max _{t_{0} \leq \theta \leq t}\left|f^{(2)}(\theta)\right|
\end{align*}
$$

Proof. The error of Eq.(15) at "actual" time $k \cdot \Delta \tau$ is evaluated by

$$
\begin{align*}
\varepsilon & =\left|D^{-q} f(k \cdot \Delta \tau)-\left(D^{-q} f(k \cdot \Delta \tau)\right)_{\text {IS }}\right| \\
& \leq \frac{T_{q}(k \cdot \Delta \tau)-T_{q}\left(t_{0}\right)}{12} \Delta T^{2} \cdot \max _{t_{0} \leq \theta \leq k \cdot \Delta \tau}\left|f^{(2)}(\theta)\right| \\
& =\frac{1}{12}\left\{\frac{(k \cdot \Delta \tau)^{q}}{\Gamma(q+1)}\right\} \Delta T^{2} \cdot \max _{t_{0} \leq \theta \leq k \cdot \Delta \tau}\left|f^{(2)}(\theta)\right| \tag{32}
\end{align*}
$$

Then the error over the entire calculation domain can be found by summation $k=1 \sim n\left(n=\left(t-t_{0}\right) / \Delta \tau\right)$ :

$$
\varepsilon \leq \sum_{k=1}^{n} \frac{1}{12}\left\{\frac{(k \cdot \Delta \tau)^{q}}{\Gamma(q+1)}\right\} \Delta T^{2} \cdot \max _{t_{0} \leq \theta \leq k \cdot \Delta \tau}\left|f^{(2)}(\theta)\right|
$$

Table 1 Calculation times of the two numerical results of $D^{ \pm 0.5} t^{2}$ for $\Delta \tau=\Delta T=0.01$.

| $t[\mathrm{~s}]$ | 5 | 10 | 20 | 50 |
| :--- | ---: | ---: | ---: | ---: |
| G1 | 0.66 | 4.92 | 37.9 | 576.6 |
| IS(FI) | 0.06 | 0.13 | 0.36 | 1.31 |
| IS(FD) | 0.25 | 0.67 | 1.90 | 7.19 |

$$
\begin{equation*}
\leq \frac{1}{12} \sum_{k=1}^{n} k^{q} \cdot \frac{\Delta \tau^{q} \cdot \Delta T^{2}}{\Gamma(q+1)} \max _{t_{0} \leq \theta \leq t}\left|f^{(2)}(\theta)\right| \tag{33}
\end{equation*}
$$

which ends the proof of the theorem.
The approximate error is estimated by Eq.(31), and the accuracy can be predicted before the calculation is performed.

## 5. Examples of Numerical Calculation

Numerical simulations were performed with the ISAlgorithm (IS) proposed in this paper and also with the conventional G1-Algorithm (G1) in order to compare the performance of the two processes. The test calculations were performed for the power function $f_{1}(t)=t^{a}$, and the sine function $f_{2}(t)=\sin \omega t$, whose analytical solutions are given by

$$
\begin{align*}
& D^{q} f_{1}(t)=D^{q} t^{a}=\frac{\Gamma(a+1)}{\Gamma(a+1-q)} t^{a-q}  \tag{34}\\
& D^{q} f_{2}(t)=D^{q} \sin \omega t=t^{-q} \sum_{n=0}^{\infty} \frac{(-1)^{n}(\omega t)^{2 n+1}}{\Gamma(2 n+2-q)} \tag{35}
\end{align*}
$$

where the parameters in the above equation were set at $q= \pm 0.5, a=2$ and $\omega=3$ for the computation.

### 5.1 Comparison of computation time for identical cases

The calculation window was set for the entire domain ( $L=t$ ) for both the IS and G1 procedures. Table 1 shows the time required to perform a computation for the $q=0.5$ of $f_{1}(t)$. The sampling period was set at $\Delta \tau(=\Delta T)=0.01[s]$ and the calculation domains were $t=5,10,20,50[s]$. The expressions for the FI and FD are exactly the same in G1, but are different in the proposed method, so are marked accordingly on the table as IS(FI) and IS(FD). The table indicates that the time required for the G1 calculation increased dramatically with increase in domain length. This increase was greatly alleviated by the new procedure. It was verified that the new procedure offers higher calculation efficiency.

## 5. 2 Assessment of calculation precision

Table 2 presents the absolute error and the estimated

Table 2 Calculation errors of the theoretical results for IS and the two numerical ones for $\Delta \tau=\Delta T=$ 0.1, 0.01.

| $D^{-0.5} t^{2}$ | $\varepsilon$ of eq. $(31)$ | IS | G1 |
| :--- | ---: | ---: | ---: |
| $\Delta T=0.1$ | 0.40 | 0.09 | 0.60 |
| $\Delta T=0.01$ | 0.04 | 0.02 | 0.36 |
| $D^{-0.5} \sin 3 t$ | $\varepsilon$ of eq. 31$)$ | IS | G1 |
| $\Delta T=0.1$ | 1.79 | 0.25 | 0.31 |
| $\Delta T=0.01$ | 0.18 | 0.04 | 0.10 |



Fig. 6 Comparison between the theoretical solution of $D^{0.5} t^{2}$ and the proposed ones for $L=1,2,3[\mathrm{~s}]$.
error according to Eq.(31) for the FI of order $q=0.5$ under both procedures for $f_{1}(t)$ and $f_{2}(t)$. Here, $t=10[s]$, the calculation window was the entire domain ( $L=t$ ), and the sampling periods were $\Delta \tau(=\Delta T)=0.1,0.01[s]$. The error of the IS computation was within the range predicted by Eq.(31) for both $f_{1}(t)$ and $f_{2}(t)$, indicating that the equation is, indeed, suitable for estimating error of IS. Furthermore, accuracy was somewhat higher for IS than for G1 for these examples. Similar results were obtained when the calculations were performed with other parameters.

## 5. 3 Influence of calculation window : $L$

Next, the calculation window $L$ was shortened in order to reduce calculation time, and the effect on calculation accuracy was examined. Figures 6 and 7 show the results of IS calculations with FD of order $q=0.5$ for $f_{1}(t)$ and $f_{2}(t)$. Parameters are set for the calculation time $t=10[s]$ and $\Delta \tau(=\Delta T)=0.01[s]$, for $f_{1}(t), L=1,2,3[s]$ and for $f_{2}(t), L=0.1,0.2,0.3[s]$. Figure 6 shows that, for a monotonically increasing function like $f_{1}(t)$, shortening $L$ causes a rapid degradation in accuracy, while a periodic function such as $f_{2}(t)$, however, even a window of $L=0.1$, which was much shorter than the period of the function, produced a fairly close approximation of the solution in Fig. 7.


Fig. 7 Comparison between the theoretical solution of $D^{0.5} \sin 3 t$ and the proposed ones for $L=$ $0.1,0.2,0.3[\mathrm{~s}]$.

## 6. Summary

This paper presents an inhomogeneous sampling algorithm (IS-Algorithm) as a method for obtaining accurate numerical solutions of fractional calculus problems at a reduced computational cost in comparison with conventional solutions, which have been severely hampered by significant increases in computational burden with increases in the length of the time domain. In the proposed method, the sampling period is set constant on a "transformed" time axis, resulting in sampling periods on the "actual" time axis whose length increases with distance into the past. This significantly reduces the number of required sample points in time.

The numerical simulations showed that the proposed procedure offers accuracy comparable to that of the conventional procedure, and a substantially superior computational efficiency. An error assessment method was also demonstrated for the proposed procedure. If the function is periodic, it has also been shown that the use of a calculation window effectively reduces computation time. Potential applications of the proposed model include practical control systems employing FDs.

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Reprinted from Trans. of the SICE
Vol. 42 No. 8 941/948 2006


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